Math 43: Spring 2020
Lecture 25 Summary

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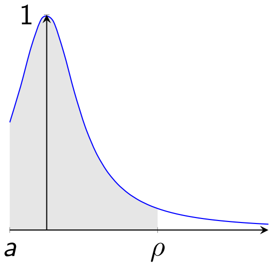
Wednesday May 27, 2020

Improper Integrals



If f is continuous on $[a, \infty)$, then we define

$$\int_a^{\infty} f(x)\,dx = \lim_{\rho\to\infty} \int_a^{\rho} f(x)\,dx.$$

Similarly,

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{\rho\to\infty} \int_{-\rho}^{a} f(x)\,dx.$$

If the limit exists, we say that the integral converges. Otherwise, we say the integral diverges. The idea is, at least in the case that in the case $f(x) \geq 0$ for all x, that the limiting value represents the area of the infinite region under the curve.

Two Sided Improper Integrals

Our contour integral methods typically involve bi-infinite integrals

$$\int_{-\infty}^{\infty} f(x)\,dx$$

for a continuous function f on $(-\infty, \infty)$. These are defined to converge only if for some $a \in (-\infty, \infty)$, both

$$\int_{-\infty}^{a} f(x)\,dx \quad \text{and} \quad \int_{a}^{\infty} f(x)\,dx$$

converge. The reason for this apparent pedantry is to make sure we are measuring something meaningful. Nevertheless, our methods lead naturally to defining the principal value

$$\text{p. v.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} f(x)\,dx$$

Some Good Stuff

Lemma

If $\int_{-\infty}^{\infty} f(x)\,dx$ *converges, then* $\mathrm{p.\,v.}\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$

Lemma (Comparison Theorem)

If $|g(x)| \leq f(x)$ *for all* x *and* $\int_{-\infty}^{\infty} f(x)\,dx < \infty$, *then* $\int_{-\infty}^{\infty} g(x)\,dx$ *converges.*

Theorem (Plus Two for Converence)

Suppose that $p(x)$ *and* $q(x)$ *are polynomials with real coefficients such that* $\deg p(x) + 2 \leq \deg q(x)$. *Then if* $q(x)$ *has no real roots,*

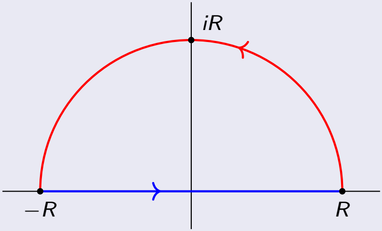
$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx$$

converges.

Improper Integrals: General Cases

Remark

Just as in Friday's lecture, we will use the contour $\Gamma_R = [-R, R] + C_R^+$ where C_R^+ is the top half of the positively oriented circle $|z| = R$ from R to $-R$.



The Basic Limit Lemma

Theorem (Basic Limit Lemma)

Let C_R^+ be the top half of the positively oriented circle $|z| = R$ from R to $-R$. Suppose that $p(z)$ and $q(z)$ are polynomials (with possibly complex coefficients) such that

$$\deg p(z) + 2 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \text{with } a \geq 0. \text{ (This means } a \in \mathbf{R}!)$$

Then

$$\lim_{R\to\infty} \int_{C_R^+} F(z)\, dz = 0.$$

The Big Reveal

Theorem (Plus Two Residue Theorem)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 2 \leq \deg q(z)$ and such that $q(z)$ has no real roots. Let $a \geq 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)\Big]$$

In particular, if $a = 0$, then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)$$

The Book

Remark

The fact that we insist that the polynomials $p(z)$ and $q(z)$ in the "Plus Two Residue Theorem" have real coefficients is crucial to the result. In the text the authors do not make that assumption, and as a result, their methods are more complicated. We will not be looking at any problems where this hypothesis is not satisfied. On the other hand, the text does not invoke the Comparison Theorem for improper integrals and has to put principal values in front to everything.

Examples

Example

$$I = \int_0^\infty \frac{x^2}{(x^2+9)^2}\,dx = 2\pi i \operatorname{Res}(F;3i) = \frac{\pi}{12}, \text{ where}$$
$$F(z) = \frac{z^2}{(z^2+9)^2}$$

Example

$$I = \int_{-\infty}^\infty \frac{\cos(x)}{x^2+2x+2}\,dx = \operatorname{Re}\big[2\pi i \operatorname{Res}(F;-1+i)\big] = \frac{\pi\cos(1)}{e}, \text{ where}$$
$$F(z) = \frac{e^{iz}}{z^2+2z+2}.$$

Example

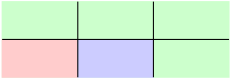
If $a, b > 0$, then
$$I = \int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx = \operatorname{Re}\big[2\pi i \operatorname{Res}(F;-ib)\big] = \frac{\pi e^{-ab}(ab+1)}{4b^3},$$
where $F(z) = \dfrac{e^{iaz}}{(z^2+b^2)^2}$.

The Final Exam

- The final exam is currently scheduled to be handed out the last day of class, Wednesday, June 3rd, and collected Friday, June 5th which is when our exam was scheduled by the Registrar.
- I am thinking of two alternate scenarios:
  1. Release the exam on Wednesday as above, but have it due by 5pm Sunday June 7th.
  2. Release the exam on Friday and still have it due at 5pm on Sunday. This makes my office hours on Thursday available for general questions.
- On Friday, I'll ask for your input to chose to keep the original plan or use one of the alternatives above.

Studying for the Final

- Like most mathematics finals, ours will be "cumulative".
- Also like most mathematics finals, the "good stuff" comes at the end of the course, so naturally we want to prove we know the "good stuff".
- Zinn's Law of Finals:



  We covered the the first two-thirds of the course on the preliminary exam and midterm. The final should cover the last third as well as the whole course. Thus Zinn's Law says that 50% of the final is on the last third of the course.
- Also, we use the material from the first two-thirds to work the material in the last third, so that material is tested implicitly.
- Good Luck.