Math 43: Spring 2020
Lecture 25 Part II

Dana P. Williams

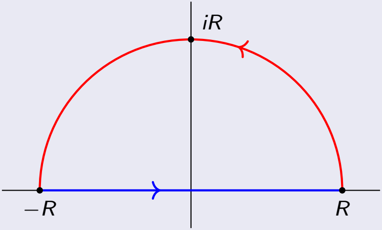
Dartmouth College

Wednesday May 27, 2020

Improper Integrals: General Cases

Remark

The point of this part of the lecture is to generalize the technique from the last part of lecture 24. We will make use of the same contour we used in that lecture: namely $\Gamma_R = [-R, R] + C_R^+$ where C_R^+ is the top half of the positively oriented circle $|z| = R$ from R to $-R$.



The Basic Limit Lemma

Theorem (Basic Limit Lemma)

Let C_R^+ be the top half of the positively oriented circle $|z| = R$ from R to $-R$. Suppose that $p(z)$ and $q(z)$ are polynomials (with possibly complex coefficients) such that

$$\deg p(z) + 2 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \text{with } a \geq 0. \text{ (This means } a \in \mathbf{R}!)$$

Then

$$\lim_{R \to \infty} \int_{C_R^+} F(z)\, dz = 0.$$

(The usefulness of the insertion of the mysterious e^{iaz} term will be explained in due course.)

Proof.

Let $n = \deg p(z)$ and $m = \deg q(z)$. Then, $n + 2 \leq m$, and as we proved in homework, if R is sufficiently large, then there are $c, d > 0$ such that $|z| \geq R$ implies

$$|p(z)| \leq c|z|^n \quad \text{and} \quad |q(z)| \geq d|z|^m \geq d|z|^{n+2}.$$

Thus for large R, for all $z \in C_R^+$ we have

$$|F(z)| \leq \frac{c \cdot R^n}{d \cdot R^{n+2}} |e^{iaz}| = \frac{c}{d} \frac{1}{R^2} |e^{iaz}|.$$

If $z \in C_R^+$, then $z = x + iy$ with $y \geq 0$! Then since $a \geq 0$,

$$|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} e^{-ay}| = e^{-ay} \leq 1.$$

Therefore if $z \in C_R^+$ for large R,

$$|F(z)| \leq \frac{c}{d} \cdot \frac{1}{R^2}$$

Finish

Proof.

Using the estimate on the last slide,

$$\left| \int_{C_R^+} F(z)\,dz \right| \leq \frac{c}{d} \cdot \frac{1}{R^2} \cdot \pi R = \frac{c\pi}{d} \cdot \frac{1}{R}.$$

Now the result follows easily from the squeeze theorem. □

The Big Reveal

Theorem (Plus Two Residue Theorem)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 2 \leq \deg q(z)$ and such that $q(z)$ has no real roots. Let $a \geq 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big]$$

In particular, if $a = 0$, then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)$$

The Proof

Proof.

Let $\Gamma_R = [-R, R] + C_R^+$ and assume that R is large enough that all the roots of $q(z)$ in the upper half-plane lie inside of Γ_R. Let

$$S = 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}(F; z).$$

This makes sense since the sum on the right has only finitely many terms corresponding to the roots of $q(z)$ in the upper half-plane! Then as in our example from the previous lecture

$$\begin{aligned} \int_{-R}^{R} F(x)\,dx &= \int_{\Gamma_R} F(z)\,dz - \int_{C_R^+} F(z)\,dz \\ &= S - \int_{C_R^+} F(z)\,dz. \end{aligned}$$

Taking the Limit

Proof.

Now we can apply the Basic Limit Lemma to conclude that

$$\mathrm{p.\,v.}\int_{-\infty}^{\infty} F(x)\,dx = S - \lim_{R\to 0}\int_{C_R^+} F(z)\,dz = S.$$

Since $p(z)$ and $q(z)$ have real coefficients, $x \in \mathbf{R}$ implies that $\frac{p(x)}{q(x)} \in \mathbf{R}$. Hence

$$\begin{aligned}\mathrm{p.\,v.}\int_{-\infty}^{\infty} F(x)\,dx &= \mathrm{p.\,v.}\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} e^{iax}\,dx \\ &= \mathrm{p.\,v.}\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\,dx + i\,\mathrm{p.\,v.}\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\,dx\end{aligned}$$

Comparison Test

Proof.

Now we can use the comparison test and our "Plus Two for Convergence" result to see that the two integrals on the right hand side converge. Hence we can get right of the principal values. Therefore

$$S = 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}(F; z) = \mathrm{p.\,v.} \int_{-\infty}^{\infty} F(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx + i \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx$$

Now we get the result by taking the real and imaginary parts of both sides . $\square$

Break Time

Remark

The fact that we insist that the polynomials $p(z)$ and $q(z)$ in the "Plus Two Residue Theorem" have real coefficients is crucial to the result. In the text the authors do not make that assumption, and as a result, their methods are more complicated. We will not be looking at any problems where this hypothesis is not satisfied. On the other hand, the text does not invoke the Comparison Theorem for improper integrals and has to put principal values in front to everything.

Now it is time for some examples. But first, . . .

. . . time for a Break.