Math 43: Spring 2020
Lecture 25 Part III

Dana P. Williams

Dartmouth College

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Plus Two Residue Theorem

Theorem (Plus Two Residue Theorem)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 2 \leq \deg q(z)$ and such that $q(z)$ has no real roots. Let $a \geq 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\,dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\,dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)\Big]$$

In particular, if $a = 0$, then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)$$

Example with $a = 0$

Example

Compute $I = \int_0^\infty \frac{x^2}{(x^2+9)^2}\,dx$.

Solution.

Since the integrand is an even function, $I = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+9)^2}\,dx$.

For better or worse, this is now just plug and chug using the Plus Two Residue Theorem. Here $a = 0$ and

$F(z) = \frac{z^2}{(z^2+9)^2} = \frac{z^2}{(z-3i)^2(z+3i)^2}$. Hence

$$I = \frac{1}{2}\cdot 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z) = \pi i \operatorname{Res}(F;3i),$$

since only $3i$ lies in the upper half-plane.

Computing the Residue

Solution Continued.

Since $3i$ is a pole of order 2,

$$\begin{aligned}\operatorname{Res}(3i) &= \lim_{z\to 3i} \frac{d}{dz}\frac{z^2}{(z+3i)^2} = \lim_{z\to 3i} \frac{2z(z+3i)^2 - 2(z+3i)z^2}{(z+3i)^4} \\ &= \lim_{z\to 3i} \frac{2z(z+3i) - 2z^2}{(z+3i)^3} = \frac{4(3i)^2 - 2(3i)^2}{8(3i)^3} = \frac{1}{12i}.\end{aligned}$$

Therefore

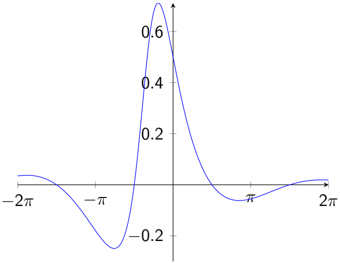
$$I = \pi i \cdot \frac{1}{12i} = \frac{\pi}{12}$$

$\square$

An Example with $a > 0$

Example

Compute $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+2x+2}\,dx.$



Here $a = 1$ and $F(z) = \dfrac{e^{iz}}{z^2+2z+2}$. (Not $F(z) = \frac{\cos(z)}{z^2+2z+2}$!!!) Then F has singularities at $-1 \pm i$. Therefore $I = \operatorname{Re}\bigl[2\pi i \operatorname{Res}(F; -1+i)\bigr]$. But by the Simple Pole Lemma,

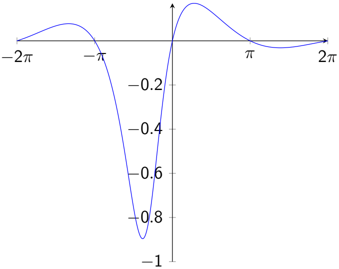
$$\operatorname{Res}(-1+i) = \frac{e^{iz}}{2z+2}\Big|_{z=-1+i} = \frac{e^{-i-1}}{2i} = \frac{1}{2ie}\bigl(\cos(1) - i\sin(1)\bigr).$$

Thus $2\pi i \operatorname{Res}(-1+i) = \frac{\pi}{e}\bigl(\cos(1) - i\sin(1)\bigr)$. Hence $I = \dfrac{\pi\cos(1)}{e}$.

While We're Here

Note that we also showed that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+2x+2}\,dx = -\frac{\pi\sin(1)}{e}.$$



One More

Example

Compute $I = \int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx$ where $a, b > 0$.

Solution.

Fortunately, the integrand is even, so $I = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(ax)}{(x^2+b^2)^2}\,dx$.

Thus we can apply our method to $F(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$. Then

$$I = \operatorname{Re}\bigl[\pi i \operatorname{Res}(F; ib)\bigr].$$

In fact, if we are paying attention, we know that the imaginary part must be 0, so we can drop the "Re", but this is not a necessary observation.

Proof.

Sadly, ib is a pole of order 2 for F. So, omitting some of the calculations,

$$\begin{aligned}\operatorname{Res}(ib) &= \lim_{z\to ib} \frac{d}{dz}\frac{e^{iaz}}{(z+ib)^2} = \lim_{z\to ib} \frac{e^{iaz}\big((ia(z+ib)-2\big)}{(z+ib)^3} \\ &= e^{-ab}\frac{(-2ab-2)}{-8ib^3} = \frac{e^{-ab}}{i}\Big(\frac{ab+1}{4b^3}\Big).\end{aligned}$$

Hence

$$I = \operatorname{Re}\Big[\pi i \frac{e^{-ab}}{i}\Big(\frac{ab+1}{4b^3}\Big)\Big] = \frac{\pi e^{-ab}(ab+1)}{4b^3}.$$

$\square$

Remark

The Plus Two Residue Theorem reduces many computations of these improper integrals to simply computing (the appropriate) residues. But the algebra can be challenging. Of course, the Plus Two Theorem does not cover all possibilities, so it is still necessary to be clever now and again.

Enough for today.