# Math 43: Spring 2020
# Lecture 26 Summary

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# Remember

## Theorem (Plus Two Residue Theorem)

*Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 2 \leq \deg q(z)$ and such that $q(z)$ has no real roots. Let $a \geq 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big]$$

*In particular, if $a = 0$, then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)$$

# Plus One

**Theorem (Plus One Residue Theorem)**

*Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 1 \leq \deg q(z)$ and such that $q(z)$ has no real roots. Let $a > 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big]$$

**Remark**

The point here is that if the cosine and sine terms are really there, then our integrals converge as principal value integrals even if the degree of $p(z)$ is only one less than the degree of $q(z)$.

# Jordan's Lemma

## Theorem (Jordan's Lemma)

*Suppose that* $p(z)$ *and* $q(z)$ *are polynomials with*

$$\deg p(z) + 1 \leq \deg q(z).$$

*Let*

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \textit{with } a > 0.$$

*Then*

$$\lim_{R \to 0} \int_{C_R^+} F(z)\, dz = 0.$$

# An Example

## Example

If $a > 0$, then we can compute

$$I = \int_{-\infty}^{\infty} \frac{x \sin(ax)}{x^2 + 2x + 1} \, dx.$$

By the Plus One Residue Theorem with $F(z) = \frac{ze^{iaz}}{z^2+2z+1}$, $I$ is

$$\begin{aligned} &= \operatorname{Im}\big(2\pi i \operatorname{Res}(F; -1+i)\big) \\ &= \frac{\pi}{e^a}\big(\cos(a) + \sin(a)\big). \end{aligned}$$

Note that our answer must be a real number. However, in this case, there is no reason that the answer must be positive.

# Back to Thinking

## Example

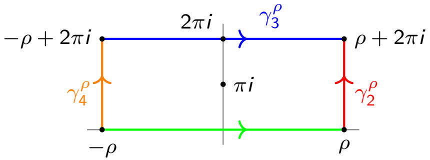
Compute $I = \text{p. v.}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$ where $0 < a < 1$.

## Remark

Let $f(z) = \frac{e^{az}}{1+e^z}$.

1. We need $a < 1$ so that $\int_0^\infty f(x)\,dx < \infty$.
2. We need $a > 0$ so that $\int_{-\infty}^0 f(x)\,dx < \infty$.
3. This means we could drop the principal value, but let's not worry about that now.
4. The real rub here is that $f$ has poles at $(2k+1)\pi i$ for all $k \in \mathbf{Z}$. Worse, there is no reason to suspect that $\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz \to 0$! Therefore using $\Gamma_R = [-R, R] + C_R^+$ as in the past won't work.

# A New Contour




We consider the positively oriented rectangular contour $\Gamma_\rho = [-\rho, \rho] + \gamma_2^\rho - \gamma_3^\rho - \gamma_4^\rho$ where the $\gamma_k^\rho$ are the directed line segments drawn to the left.

The choice of orientations are just to ease some computations down the road. For now note that for all $\rho > 0$, the Cauchy Residue Theorem implies

$$\int_{\Gamma_\rho} f(z)\, dz = 2\pi i \operatorname{Res}(f; i\pi).$$

# Compute

We had to show that $\lim_{\rho\to\infty}\int_{\gamma_k^\rho} f(z)\,dz = 0$ for the two vertical line segments, while

$$\lim_{\rho\to\infty}\int_{\gamma_3^\rho} f(z)\,dz = e^{2\pi i a}\,\mathrm{p.\,v.}\int_{-\infty}^{\infty} f(x)\,dx.$$

Putting it all together gives

$$\begin{aligned}\mathrm{p.\,v.}\int_{-\infty}^{\infty} f(x)\,dx &= \frac{2\pi i}{1-e^{2\pi i a}}\operatorname{Res}(f;\pi i)\\ &= \frac{\pi}{\sin(\pi a)}.\end{aligned}$$

Problem #11 in §6.3 is another fun variation on this theme.

# Quite Enough

**Remark**

That completes our investigations into evaluating real integrals using residues. We'll end the course Monday and Wednesday next week with some more applications of the residue theory.