Math 43: Spring 2020
Lecture 26 Part I

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Remember

Theorem (Plus Two Residue Theorem)

Suppose that $p(z)$ and $q(z)$ are polynomials with real coefficients such that $\deg p(z) + 2 \le \deg q(z)$ and such that $q(z)$ has no real roots. Let $a \ge 0$ and define $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\,dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\,dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)\Big]$$

In particular, if $a = 0$, then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F;z)$$

Plus One

Theorem (Plus One Residue Theorem)

Suppose that $p(z)$ *and* $q(z)$ *are polynomials with real coefficients such that* $\deg p(z) + 1 \leq \deg q(z)$ *and such that* $q(z)$ *has no real roots. Let* $a > 0$ *and define* $F(z) = \frac{p(z)}{q(z)} e^{iaz}$. *Then*

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \cos(ax)\, dx = \operatorname{Re}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big] \quad \textit{and}$$

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} \sin(ax)\, dx = \operatorname{Im}\Big[2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res}(F; z)\Big]$$

Remark

The point here is that if the cosine and sine terms are really there, then our integrals converge as principal value integrals even if the degree of $p(z)$ is only one less than the degree of $q(z)$.

The Proof

Remark (Fixing The Proof)

Reviewing the proof of the "Plus Two Residue Theorem", the plus two condition on the degrees of $p(z)$ and $q(z)$ is only used twice. One to prove the Basic Limit Lemma and secondly to ensure that the integrals in question converge. Hence if we are willing to put the principal values back in, then all we have to do is replace the hypothesis that "$\deg p(z) + 2 \leq \deg q(z)$ and $a \geq 0$" in the Basic Limit Lemma with "$\deg p(z) + 1 \leq \deg q(z)$ and $a > 0$". Then the proof of the Plus Two Residue Theorem suffices to prove the Plus One Residue Theorem. As it turns out, we can still get rid of the principal values, but we will omit the proof of that.

Jordan's Lemma

Theorem (Jordan's Lemma)

Suppose that $p(z)$ and $q(z)$ are polynomials with

$$\deg p(z) + 1 \leq \deg q(z).$$

Let

$$F(z) = \frac{p(z)}{q(z)} e^{iaz} \quad \textit{with } a > 0.$$

Then

$$\lim_{R \to 0} \int_{C_R^+} F(z)\, dz = 0.$$

False Start

Proof.

Just as in the proof of the Basic Limit Lemma, for R sufficiently large and $z \in C_R^+$, we have

$$\left|\frac{p(z)}{q(z)}\right| \le \frac{K}{R}$$

for some constant K. (Before with had an R^2 in place of R.) Since we still have $|e^{iaz}| \le 1$ if $\mathrm{Im}(z) \ge 0$ and $a > 0$, we get the crude estimate that

$$\left|\int_{C_R^+} F(z)\,dz\right| \le \frac{K}{R} \cdot 1 \cdot \pi R = K\pi.$$

Of course, this is not very helpful. Instead, we have to roll up our sleeves and get down and dirty

Getting serious

Proof.

We parameterize C_R^+ by $z(t) = Re^{it}$ for $t \in [0, \pi]$. Then

$$\begin{aligned}
\left| \int_{C_R^+} F(z)\,dz \right| &= \left| \int_0^\pi \frac{p(z(t))}{q(z(t))} e^{iaz(t)} z'(t)\,dt \right| \\
&\le \int_0^\pi \left| \frac{p(z(t))}{q(z(t))} e^{iaz(t)} z'(t) \right| dt \\
&= \int_0^\pi \left| \frac{p(z(t))}{q(z(t))} \right| \left| e^{iaz(t)} \right| |z'(t)|\,dt \quad (\ddagger)
\end{aligned}$$

▸ Return

Estimates

Proof.

Since $z \in C_R^+$, $\left|\dfrac{p(z(t))}{q(z(t))}\right| \le \dfrac{K}{R}$ for large R. While

$$\begin{aligned}\left|e^{iaz(t)}\right| = \left|e^{iaRe^{it}}\right| = \left|e^{iaR(\cos(t)+i\sin(t))}\right| \\ = e^{-aR\sin(t)}.\end{aligned}$$

And $|z'(t)| = |Rie^{it}| = R$. Then plugging into $(\ddagger)$ Last Slide, we have

$$\left|\int_{C_R^+} F(z)\,dz\right| \le K\int_0^\pi e^{-aR\sin(t)}\,dt.$$

Almost There

Proof.

Now

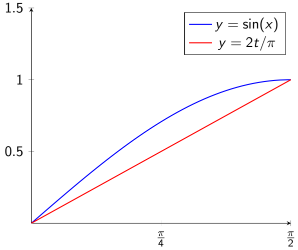
$$K \int_0^{\pi} e^{-aR\sin(t)}\,dt = K \int_0^{\frac{\pi}{2}} e^{-aR\sin(t)}\,dt + K \int_{\frac{\pi}{2}}^{\pi} e^{-aR\sin(t)}\,dt$$

which, after substituting $u = \pi - t$ in the second integral and since $\sin(\pi - t) = \sin(t)$, becomes

$$= 2K \int_0^{\frac{\pi}{2}} e^{-aR\sin(t)}\,dt$$

$\square$

Finish



Proof.

On the interval $[0, \frac{\pi}{2}]$ we have $\sin(t) \geq \frac{2}{\pi}t$. Therefore

$$2K \int_0^{\frac{\pi}{2}} e^{-aR\sin(t)}\,dt \leq 2K \int_0^{\frac{\pi}{2}} e^{-\frac{aR2t}{\pi}}\,dt$$

But the last integral is a routine substitution and equal to

$$\leq 2K\left(\frac{-\pi}{aR2}\right)e^{-\frac{aR2t}{\pi}}\Big|_{t=0}^{t=\frac{\pi}{2}} = \frac{2K\pi}{aR2}\left(1 - e^{-aR}\right).$$

However this expression tends to 0 as $R \to \infty$. This completes the proof of Jordan's Lemma and hence of the Plus One Residue Theorem as well.

Break Time

Remark

Of course, now we need an example to make all this work worthwhile. But first ...

...time for a Break.