Math 43: Spring 2020
Lecture 26 Part III

Dana P. Williams

Dartmouth College

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Back to Thinking

Example

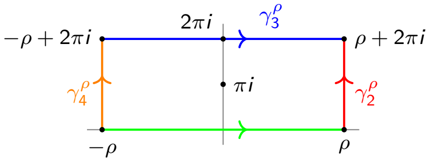
Compute $I = \text{p.v.}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$ where $0 < a < 1$.

Remark

Let $f(z) = \frac{e^{az}}{1+e^z}$.

1. We need $a < 1$ so that $\int_0^\infty f(x)\,dx < \infty$.
2. We need $a > 0$ so that $\int_{-\infty}^0 f(x)\,dx < \infty$.
3. This means we could drop the principal value, but let's not worry about that now.
4. The real rub here is that f has poles at $(2k+1)\pi i$ for all $k \in \mathbf{Z}$. Worse, there is no reason to suspect that $\lim_{R\to\infty} \int_{C_R^+} f(z)\,dz \to 0$! Therefore using $\Gamma_R = [-R, R] + C_R^+$ as in the past won't work.

A New Contour



We consider the positively oriented rectangular contour $\Gamma_\rho = [-\rho, \rho] + \gamma_2^\rho - \gamma_3^\rho - \gamma_4^\rho$ where the γ_k^ρ are the directed line segments drawn to the left.

The choice of orientations are just to ease some computations down the road. For now note that for all $\rho > 0$, the Cauchy Residue Theorem implies

$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i \operatorname{Res}(f; i\pi).$$

Of course

$$\int_{[-\rho,\rho]} f(z)\,dz = \int_{-\rho}^{\rho} f(t)\,dt.$$

Now we parameterize γ_3^ρ by $z(t) = t + 2\pi i$ for $t \in [-\rho, \rho]$. Then

$$\begin{aligned}\int_{\gamma_3^\rho} f(z)\,dz &= \int_{-\rho}^{\rho} \frac{e^{az(t)}}{1+e^{z(t)}} z'(t)\,dt \\ &= \int_{-\rho}^{\rho} \frac{e^{2\pi ia}e^{at}}{1+e^{t}} \cdot (1)\,dt \\ &= e^{2\pi ia}\int_{-\rho}^{\rho} f(t)\,dt.\end{aligned}$$

Putting Things Together

We have

$$\begin{aligned}2\pi i \operatorname{Res}(f;\pi i) &= \int_{\Gamma_\rho} f(z)\,dz \\ &= \int_{[-\rho,\rho]} f(z)\,dz + \int_{\gamma_2^\rho} f(z)\,dz - \int_{\gamma_3^\rho} f(z)\,dz - \int_{\gamma_4^\rho} f(z)\,dz \\ &= (1-e^{2\pi i a})\int_{-\rho}^{\rho} f(x)\,dx + \int_{\gamma_2^\rho} f(z)\,dx - \int_{\gamma_4^\rho} f(z)\,dz.\end{aligned}$$

Or

$$(1-e^{2\pi i a})\int_{-\rho}^{\rho} f(x)\,dx = 2\pi i \operatorname{Res}(f;\pi i) - \int_{\gamma_2^\rho} f(z)\,dz + \int_{\gamma_4^\rho} f(z)\,dz.$$

Therefore if we can show $\lim_{\rho\to\infty}\int_{\gamma_k^\rho} f(z)\,dz = 0$ for $k = 2, 4$, then

$$\text{p. v.}\int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi i}{1-e^{2\pi i a}}\operatorname{Res}(f;\pi i).$$

Estimating

We can parameterize γ_2^ρ by $z(t) = \rho + it$ for $t \in [0, 2\pi]$. Then

$$\begin{aligned}\left|f(z(t))\right| = \left|\frac{e^{a(\rho+it)}}{1+e^{\rho+it}}\right| &= \frac{e^{a\rho}}{\left|1+e^{\rho}e^{it}\right|} \\ &\leq \frac{e^{a\rho}}{e^{\rho}-1}.\end{aligned}$$

Therefore

$$\left|\int_{\gamma_2^\rho} f(z)\,dz\right| \leq \frac{e^{a\rho}}{e^{\rho}-1}\cdot 2\pi.$$

Since $0 < a < 1$, the right-hand side goes to zero as $\rho \to \infty$. I'll leave the estimate for the integral over γ_4^ρ as a homework exercise.

The Residue

By the Simple Pole Lemma,

$$\operatorname{Res}(f;\pi i) = \frac{e^{az}}{e^{z}}\Big|_{z=\pi i} = -e^{i\pi a}.$$

Hence

$$I = \frac{-2\pi i e^{i\pi a}}{1-e^{2\pi i a}}$$

which, since the answer must be a real number is unacceptable. But notice that this is equal to

$$= \frac{-2\pi i}{e^{-i\pi a}-e^{i\pi a}} = \pi\frac{2i}{e^{i\pi a}-e^{-i\pi a}} = \frac{\pi}{\sin(\pi a)}.$$

I hope you enjoyed that.

Quite Enough

Remark

There are many more variations we could explore, but I think you may have reached your limit (so to speak). We'll end the course Monday and Wednesday next week with some more applications of the residue theory

But that is enough lecture for one day.