Math 43: Spring 2020
Lecture 27 Summary

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Monday June 1, 2020

Today

1. We should be recording.
2. The Final Exam will be made available Wednesday after our last class meeting and must be uploaded by SUNDAY at 9pm.
3. Extra office hours today from 1:30-2:30.
4. Office hours as usual TTh.

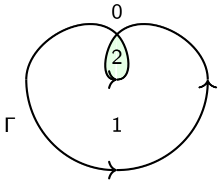
The Index

Definition

Let Γ be a (not necessarily simple) closed contour. If $a \notin \Gamma$, then we call

$$\operatorname{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-a}\, dz$$

the index of Γ about a.



The idea is simple. The index is supposed to count the number of times Γ encircles a in a counterclockwise direction. In simple examples, such as at left, this is borne out by the Deformation Invariance Theorem.

Getting Precise

Theorem

Let Γ be a closed contour. If $a \notin \Gamma$, then $\operatorname{Ind}_\Gamma(a)$ is an integer. (Thus what every $\operatorname{Ind}_\Gamma$ "counts", at least it counts it in whole numbers!)

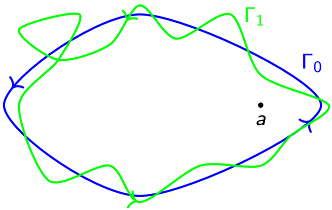
Walking the Dog

Theorem (Walking the Dog Lemma)

Suppose that Γ_0 and Γ_1 are closed contours with admissible parameterizations $z_k : [0,1] \to \mathbf{C}$ for $k = 0$ and $k = 1$. If $a \in \mathbf{C}$ is such that

$$\left|z_0(t) - z_1(t)\right| < \left|z_0(t) - a\right| \quad \text{for all } t \in [0,1],$$

then $\operatorname{Ind}_{\Gamma_0}(a) = \operatorname{Ind}_{\Gamma_1}(a)$.



The idea is that if Willy and I take a walk around the bonfire on the Green and Willy never gets further from me than I am from the bondfire, then we walk around the bonfire the same number of times.

The Value of Homework

As a corollary of the Cauchy Residue Theorem, we proved the following as a homework assignment.

Theorem (HW EP-2)

Suppose that f is analytic on and inside a positively oriented simple closed contour Γ and that f does not vanish on Γ. Then f has at most finitely many zeros inside of Γ and

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = N_f$$

where N_f is the number of zeros of f inside Γ counted up to multiplicity.

Finitely Many Zeros

Remark

On homework, we were allowed to assume that f had only finitely many zeros inside Γ. But as we argued in lecture, this is automatics since the zeros on non-constant analytic functions are isolated.

Remark

If Γ is a contour in a domain D and f is analytic on D, then $f(\Gamma)$ is also a contour. If $z : [0,1] \to \mathbf{C}$ is an admissible paramterization of Γ, then $w : [0,1] \to \mathbf{C}$, given by $w(t) = f(z(t))$ is an admissible parameterization of $f(\Gamma)$.

Homework Upgrade

Theorem

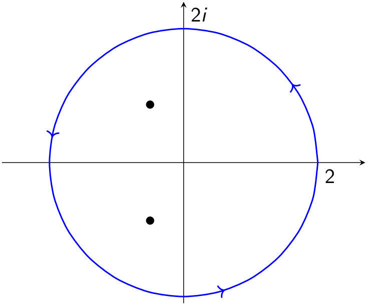
Suppose that f is analytic on and inside a positively oriented simple closed contour Γ and that f does not vanish on Γ. Then

$$\operatorname{Ind}_{f(\Gamma)}(0) = N_f$$

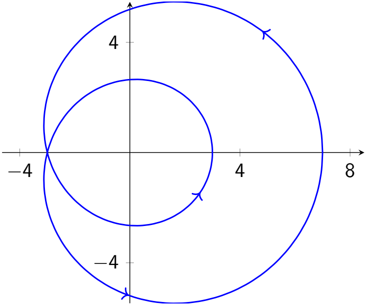
where N_f is the number of zeros of f inside of Γ counted up to multiplicity.

An Example

The function $f(z) = z^2 + z + 1$ has exactly two simple zeros inside the positively oriented circle Γ equal to $|z| = 2$ and is nonzero on Γ. We can verify the previous result as follows.



(a) The circle Γ of radius 2 centered at the origin and the zeros of $f(z) = z^2 + z + 1$



(b) The composite path $f(\Gamma)$ with $\mathrm{Ind}_{f(\Gamma)}(0) = 2$

Figure: Counting the zeros of $f(z) = z^2 + z + 1$

Rouche's Theorem

Theorem (Rouche's Theorem)

Suppose that f and g are analytic on and inside a simple closed contour Γ and that

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \Gamma.$$

Then f and g have the same number of zeros inside of Γ up to multiplicity.

An Example

Example

Show that $p(z) = z^5 + 3z^3 + 7$ has five distinct zeros in the disk $B_2(0)$.

Finishing Up

We will finish up on Wednesday with one more key result about analytic functions. Our class meeting will be the last opportunity for questions on the material in the course.