Math 43: Spring 2020
Lecture 27 Part I

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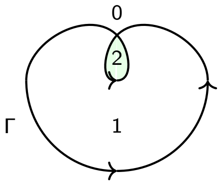
Definition

Let Γ be a (not necessarily simple) closed contour. If $a \notin \Gamma$, then we call

$$\operatorname{Ind}_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-a}\, dz$$

the index of Γ about a.



Using the Deformation Invariance Theorem as motivation, it would seem that $\operatorname{Ind}_\Gamma(a)$ should "count" the number of times Γ wraps around a in a counter-clockwise direction. In the example at left, it is clear that $\operatorname{Ind}_\Gamma(a)$ equals two if a is in the small green domain, equals one if a is in the middle domain, and is always zero in the domain which is the exterior of Γ.

Getting Precise

Theorem

Let Γ be a closed contour. If $a \notin \Gamma$, then $\operatorname{Ind}_\Gamma(a)$ is an integer. (Thus what every $\operatorname{Ind}_\Gamma$ "counts", at least it counts it in whole numbers!)

Proof.

Let $z : [0,1] \to \mathbf{C}$ be an admissible parameterization of Γ. Then

$$2\pi i \operatorname{Ind}_\Gamma(a) = \int_0^1 \frac{z'(t)}{z(t)-a}\,dt.$$

For $s \in [0,1]$, let

$$\varphi(s) = \exp\Bigl(\int_0^s \frac{z'(t)}{z(t)-a}\,dt\Bigr).$$

It will suffice to see that $\varphi(1) = 1$.

Proof

Proof.

Let $\psi(t) = \dfrac{\varphi(t)}{z(t) - a}$. Then since $\varphi'(t) = \left(\frac{z'(t)}{z(t)-a}\right)\varphi(t)$, we have

$$\begin{aligned}
\psi'(t) &= \frac{\varphi'(t)(z(t)-a) - z'(t)\varphi(t)}{(z(t)-a)^2} \\
&= \frac{\left(\frac{z'(t)}{z(t)-a}\right)\varphi(t)(z(t)-a) - z'(t)\varphi(t)}{(z(t)-a)^2} \\
&= 0!
\end{aligned}$$

Therefore ψ is constant.

Proof

Proof.

Since ψ is constant,

$$\psi(t) = \frac{\varphi(t)}{z(t) - a} = \psi(0) = \frac{1}{z(0) - a}.$$

Thus

$$\varphi(t) = \frac{z(t) - a}{z(0) - a}.$$

Since $z(1) = z(0)$, $\varphi(1) = 1$ as required. $\square$

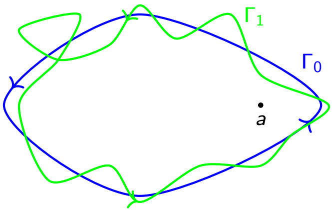
Walking the Dog

Theorem (Walking the Dog Lemma)

Suppose that Γ_0 and Γ_1 are closed contours with admissible parameterizations $z_k : [0,1] \to \mathbf{C}$ for $k = 0$ and $k = 1$. If $a \in \mathbf{C}$ is such that

$$\big|z_0(t) - z_1(t)\big| < \big|z_0(t) - a\big| \quad \textit{for all } t \in [0,1],$$

then $\mathrm{Ind}_{\Gamma_0}(a) = \mathrm{Ind}_{\Gamma_1}(a)$.



The idea is that if Willy and I take a walk around the bonfire on the Green and Willy never gets further from me than I am from the bondfire, then we walk around the bonfire the same number of times.

Proof.

We can parameterize a new contour, Γ, by

$$z(t) = \frac{z_1(t) - a}{z_0(t) - a} \quad \text{with } t \in [0,1].$$

Notice that

$$\begin{aligned} \left| z(t) - 1 \right| &= \left| \frac{z_1(t) - a - (z_0(t) - a)}{z_0(t) - a} \right| \\ &= \left| \frac{z_1(t) - z_0(t)}{z_0(t) - a} \right| < 1. \end{aligned}$$

This means Γ lies inside the simply connected disk $B_1(1)$. Thus by the Cauchy Integral Theorem

$$\operatorname{Ind}_\Gamma(0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{w}\, dw = 0.$$

Finish

Proof.

On the other hand,

$$\begin{aligned}
\mathrm{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_0^1 \frac{1}{z(t)} z'(t)\,dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{z_0(t)-a}{z_1(t)-a}\cdot\frac{z_1'(t)(z_0(t)-a)-z_0'(t)(z_1(t)-a)}{(z_0(t)-a)^2}\,dt \\
&= \frac{1}{2\pi i}\int_0^1 \Big(\frac{z_1'(t)}{z_1(t)-a}-\frac{z_0'(t)}{z_0(t)-a}\Big)\,dt \\
&= \mathrm{Ind}_{\Gamma_1}(a)-\mathrm{Ind}_{\Gamma_0}(a).
\end{aligned}$$

Since $\mathrm{Ind}_\Gamma(0)=0$, we're done. □

Break Time

Remark

Ok, so we've walked the dog.

Now we get a break too.