Math 43: Spring 2020
Lecture 27 Part II

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The Value of Homework

As a corollary of the Cauchy Residue Theorem, we proved the following as a homework assignment.

Theorem (HW EP-2)

Suppose that f is analytic on and inside a positively oriented simple closed contour Γ and that f does not vanish on Γ. Then f has at most finitely many zeros inside of Γ and

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)}{f(z)}\,dz = N_f$$

where N_f is the number of zeros of f inside Γ counted up to multiplicity.

Finitely Many Zeros

You proved the result on the previous slide using the assumption that f had finitely many zeros inside of Γ. To see that this must be the case, suppose f had infinitely many zeros in the interior, D, of Γ. Because $D \cup \Gamma$ is closed and bounded, there would be a sequence (z_k) of zeros of f converging to some $z_0 \in D \cup \Gamma$. Since f does not vanish on Γ, we must have $z_0 \in D$. By continuity, $f(z_0) = \lim_k f(z_k) = 0$. But then z_0 is not an isolated zero of f and f must be constantly equal to zero. But then f is zero on Γ as well. So the assumption that f has only finitely many zeros inside of Γ is automatically satisfied.

But You Made Me Learn About the Index?!?

Let f and Γ be in the homework result. If $z : [0,1] \to \mathbf{C}$ is an admissible parameterization of Γ, then $w : [0,1] \to \mathbf{C}$ given by $w(t) = f(z(t))$ is an admissible parameterization of $f(\Gamma)$. In fact, $w'(t) = f'(z(t))z'(t)$. Then

$$\begin{aligned} N_f = \frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i} \int_0^1 \frac{f'(z(t))z'(t)}{f(z(t))}\,dt \\ &= \frac{1}{2\pi i} \int_0^1 \frac{w'(t)}{w(t)}\,dt \\ &= \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{1}{w}\,dw \\ &= \mathrm{Ind}_{f(\Gamma)}(0). \end{aligned}$$

Homework Upgrade

Theorem

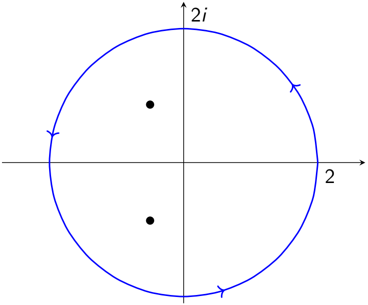
Suppose that f is analytic on and inside a positively oriented simple closed contour Γ and that f does not vanish on Γ. Then

$$\operatorname{Ind}_{f(\Gamma)}(0) = N_f$$

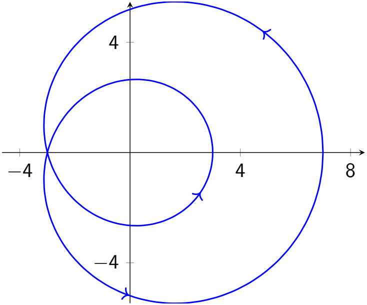
where N_f is the number of zeros of f inside of Γ counted up to multiplicity.

An Example

The function $f(z) = z^2 + z + 1$ has exactly two simple zeros inside the positively oriented circle Γ equal to $|z| = 2$ and is nonzero on Γ. We can verify the previous result as follows.



(a) The circle Γ of radius 2 centered at the origin and the zeros of $f(z) = z^2 + z + 1$



(b) The composite path $f(\Gamma)$ with $\mathrm{Ind}_{f(\Gamma)}(0) = 2$

Figure: Counting the zeros of $f(z) = z^2 + z + 1$

Rouche's Theorem

Theorem (Rouche's Theorem)

Suppose that f and g are analytic on and inside a simple closed contour Γ and that

$$|f(z) - g(z)| < |f(z)| \quad \text{for all } z \in \Gamma. \tag{‡}$$

Then f and g have the same number of zeros inside of Γ up to multiplicity.

Proof.

Note that (‡) implies that Neither f nor g vanishes on Γ. Let $z : [0, 1] \to \mathbf{C}$ be an admissible parameterization of Γ.

Proof.

Note that

$$\left|f(z(t)) - g(z(t))\right| < \left|f(z(t)) - 0\right| \quad \text{for all } t \in [0,1].$$

Now we can apply the Walking the Dog Lemma to $f(\Gamma)$ and $g(\Gamma)$ so that $\mathrm{Ind}_{f(\Gamma)}(0) = \mathrm{Ind}_{g(\Gamma)}(0)$. Then

$$N_f = \mathrm{Ind}_{f(\Gamma)}(0) = \mathrm{Ind}_{g(\Gamma)}(0) = N_g. \qquad \square$$

An Example

Example

Show that $p(z) = z^5 + 3z^3 + 7$ has five distinct zeros in the disk $B_2(0)$.

Solution.

Let $f(z) = z^5$. Then if $|z| = 2$, then

$$|f(z) - p(z)| = |3z^3 + 7| \leq 24 + 7 = 31 < 32 = |f(z)|.$$

Since, up to multiplicity, $f(z) = z^5$ has 5 zeros inside $|z| = 2$, by Rouche's Theorem, so does $p(z)$. The only remaining question is whether these zeros are distinct. But $p'(z) = 5z^4 + 9z^2 = z^2(5z^2 + 9)$. It is easy to check, by plugging in, that none of the zeros of $p'(z)$ are also zeros of $p(z)$. Thus all the zeros of $p(z)$ are simple and must be distinct. $\square$

Break Time

Remark

Alright, time to rest a bit. We have about a half a lecture to go. We'll save that for Wednesday.

Enough