Math 43: Spring 2020
Lecture 28 Part I

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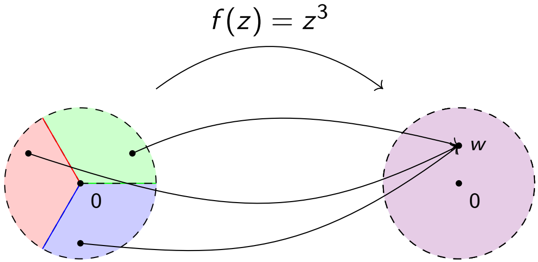
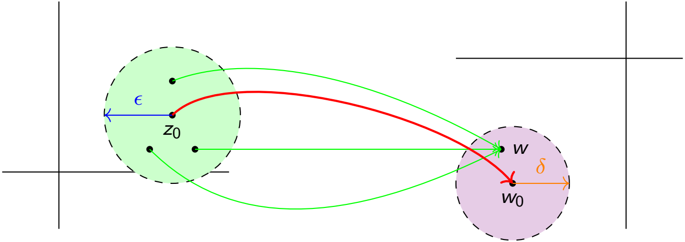


We want to look at the behavior of the map $f(z) = z^3$ from the disk $|z| < 1$ to itself. Note that f has a zero of order 3 at $z = 0$. Also, f maps the green sector $S_1 = \{\, re^{i\theta} : 0 \leq r < 1 \text{ and } 0 \leq \theta < \frac{2\pi}{3} \,\}$ onto $|z| < 1$. But f also maps the red sector $S_2 = \{\, re^{i\theta} : 0 \leq r < 1 \text{ and } \frac{2\pi}{3} \leq \theta < \frac{4\pi}{3} \,\}$ onto $|z| < 1$, the sector $S_3 = \{\, re^{i\theta} : 0 \leq r < 1 \text{ and } \frac{4\pi}{3} \leq \theta < 2\pi \,\}$ onto $|z| < 1$. Thus we get a "triple covering" of the punctured disk $B_1'(0)$.

Theorem (Local Behavior)

Suppose that f is analytic and non-constant in a domain D. Suppose that w_0 is in the range of f so that $g_{w_0}(z) = f(z) - w_0$ has a zero of order $m \geq 1$ at z_0. Then there is a $\epsilon > 0$ and a $\delta > 0$ such that $f(z) = w$ has exactly m distinct solutions in $B'_\epsilon(z_0) \subset D$ whenever $w \in B'_\delta(w_0)$.

Picture for $m = 3$:



The Proof

Proof.

Since the zeros of non-constant analytic functions are isolated and D is open, there is a $\epsilon > 0$ such that

1. $B_{2\epsilon}(z_0) \subset D$,
2. g_{w_0} has no zeros in $B'_{2\epsilon}(z_0)$, and
3. $f' = g'_{w_0}$ has no zeros in $B'_{2\epsilon}(z_0)$.

Let Γ_ϵ be the positively oriented circle $|z - z_0| = \epsilon$. Let

$$\delta = \min_{z \in \Gamma_\epsilon} |g_{w_0}(z)|.$$

Then $\delta > 0$ by item 2 above.

More of the Proof

Proof.

If $w \in B'_\delta(w_0)$, then let $g_w(z) = f(z) - w$. Then if $z \in \Gamma_\epsilon$, we have

$$|g_{w_0}(z) - g_w(z)| = |w_0 - w| < \delta \le |g_{w_0}(z)|.$$

By Rouche's Theorem, g_{w_0} and g_w have the same number of zeros (up to multiplicity) inside Γ_ϵ. Since g_{w_0} has m zeros inside $|z - z_0| = \epsilon$, g_w must have m zeros in $B'_\epsilon(z_0)$. Since $g'_w = f'$ never vanishes in $B'_\epsilon(z_0)$ by item 3, all the zeros of g_w must be distinct. □

One-To-One Functions

Theorem

Suppose that f is analytic on D and that there is a $z_0 \in D$ such that $f'(z_0) = 0$. Then f is not one-to-one on D. Hence if f is one-to-one on D, then $f'(z) \neq 0$ for all $z \in D$.

Proof.

We can assume that f is non-constant as the conclusion is obvious otherwise. Let $w_0 = f(z_0)$. Then $g_{w_0}(z) = f(z) - w_0$ has a zero of order $m \geq 2$ at z_0. Then by the previous result, there is a $\epsilon > 0$ and a $\delta >)$ such that $B_\epsilon(z_0) \subset D$ and such that given $w \in B'_\delta(z_0)$ there are m distinct elements $z_1, \dots, z_m$ in $B'_\epsilon(z_0)$ such that $f(z_k) = w$. Since $m \geq 2$, f is not one-to-one. This proves the first assertion and the second assertion follows from the first. $\square$

Remark

The converse to the previous theorem is false. Consider $f(z) = e^z$.

The Open Mapping Theorem

Theorem (Open Mapping Theorem)

Suppose that f is a non-constant analytic function on a domain D. Then the range of f, $f(D) = \{ f(z) : z \in D \}$, is open in $\mathbf{C}$.

Proof.

Let $w_0 \in f(D)$. We need to find $\delta > 0$ such that $B_\delta(w_0) \subset f(D)$. Let $z_0 \in D$ be such that $f(z_0) = w_0$. Since $g_{w_0}(z) = f(z) - w_0$ has a zero of order $m \geq 1$ at z_0, there are positive ϵ and δ such that for all $w \in B'_\delta(w_0)$ there is at least one $z \in B'_\epsilon(z_0)$ such that $f(z) = w$. But then

$$B_\delta(w_0) \subset f(B_\epsilon(z_0)) \subset f(D).$$

That completes the proof. □

A Corollary

We will explore some consequences of the Open Mapping Theorem in homework. But here is a short proof of the Maximum Modulus Theorem (with a slight upgrade).

Theorem (Maximum Modulus Principle)

Suppose that f is analytic on a domain D. If $|f(z)|$ has a local maximum on D, then f is constant.

Proof.

Suppose that f is not constant, but yet there is a $\epsilon > 0$ such that $|f(z)| \le |f(z_0)|$ for all $z \in B_\epsilon(z_0)$. Since f is non-constant, $f(B_\epsilon(z_0))$ is open. Hence there is a $\delta > 0$ such that $B_\delta(f(z_0)) \subset f(B_\epsilon(z_0))$. If $f(z_0) = |f(z_0)|e^{i\theta_0}$, then $w = \big(|f(z_0)| + \frac{\delta}{2}\big)e^{i\theta_0} = f(z_0) + \frac{\delta}{2}e^{i\theta_0} \in B_\delta(f(z_0)) \subset f(B_\epsilon(z_0))$. Thus there is a $z \in B_\epsilon(z_0)$ such that $f(z) = w$ and $|f(z)| = |w| = |f(z_0)| + \frac{\delta}{2} > |f(z_0)|$. This contradicts our assumptions. $\square$

Closer to Little Picard

Corollary

Suppose that f is a non-constant entire function. Then its range, $f(\mathbf{C})$, is open and dense.

Proof.

We already proved (on the midterm) that the range of f is dense. It is also open by the Open Mapping Theorem. $\square$

All Good Things . . .

Remark

Our Complex Journey has come to an end. While this wasn't the way any of us envisioned spring term for 2020, I hope that you enjoyed the journey nevertheless.

Have a great summer and I hope to meet some of you in person in the not too distant future.