Math 43: Spring 2020
Lecture 3 Part 1

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n^{th} Roots

Remark

Using the polar form $z = re^{i\theta}$, it is easy to work with integral powers: $z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$. Now we want to reverse the process. We want to solve for w such that $w^n = z$.

Theorem

Suppose that z is a nonzero complex number with polar form $z = re^{i\theta}$. If $n \in \mathbf{N}$, then z has n distinct n^{th}-roots given by

$$w_k = \sqrt[n]{r}\exp\left(i\frac{\theta + 2\pi k}{n}\right) \quad \textit{for } k = 0, 1, 2, \dots, n-1.$$

Remark

This is remarkable. In some sense, we invented the complex numbers to provide square roots of negative numbers. By introducing i into the mix, we have somehow arranged that all complex numbers have n distinct n^{th}-roots!

The Proof

Proof.

Clearly, $w_k^n = z$. Now suppose that $w = \rho e^{i\varphi}$ is such that $w^n = z$. Then $\rho^n e^{in\varphi} = re^{i\theta}$. Then $\rho = \sqrt[n]{r}$. Also $n\varphi = \theta + 2\pi m$ for $m \in \mathbf{Z}$. Therefore

$$\varphi = \frac{\theta}{n} + \frac{2\pi m}{n}.$$

and

$$w = w_m = \sqrt[n]{r}\exp\Bigl(i\frac{\theta + 2\pi m}{n}\Bigr).$$

But if $m = ns + k$ with $0 \leq k < n$, then $w_m = w_k$. Thus the w_k are exactly the n^{th}-roots of z. □

Roots of 1

Example

Find all (the complex) cube roots of 1.

Solution.

Since we are working over the complex numbers, there will be 3!
Since $1 = 1 \cdot e^{i \cdot 0}$, the cube roots are

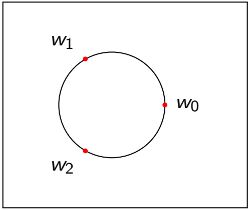
$$w_k = \exp\Big(i\frac{0 + 2\pi k}{3}\Big) \quad \text{for } k = 0, 1, 2.$$

Then

$$w_0 = 1, \quad w_1 = \exp\Big(i\frac{2\pi}{3}\Big) = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$\text{and} \quad w_2 = \exp\Big(\frac{4\pi}{3}\Big) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \qquad \square$$

Note that $\{ w_0, w_1, w_2 \} = \{ 1, w_1, w_1^2 \}$.
We get a similar
picture in general for the n^{th}-roots of 1.
They are of the form $\{ 1, w_1, w_1^2, \dots, w_1^{n-1} \}$
where $w_1 = \exp\left(i\frac{2\pi}{n}\right)$. You may remember
drawing a picture of this back when we
wrote $w_1 = \text{cis}\left(\frac{2\pi}{n}\right)$ in place of $w_1 = e^{i\frac{2\pi}{n}}$.



A Not So Trivial Example

Example

Find $(1 - i\sqrt{3})^{\frac{1}{2}}$.

Solution.

Note that there will be exactly two answers! Since $1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}$ the square roots are

$$w_k = \sqrt{2}\exp\Big(\frac{-\frac{\pi}{3} + 2\pi k}{2}\Big) \quad \text{for } k = 0, 1.$$

Thus $w_0 = \sqrt{2}e^{-\frac{\pi}{6}} = \boxed{\sqrt{2}\big(\frac{\sqrt{3}}{2} - i\frac{1}{2}\big)}$ and $w_1 = \sqrt{2}e^{i\frac{5\pi}{6}} = \boxed{\sqrt{2}\big(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\big)}$. □

No Square Root Function

Remark (Important)

In the previous, example $(1 - i\sqrt{3})^{\frac{1}{2}} = \{ w_0, w_1 \}$. Of course, $w_1 = -w_0$. If we had paused to think, we would have realized this going in! We didn't have to compute both. But notice that there is no preferred choice of square root. Hence we never write something like $\sqrt{1 - i\sqrt{3}}$. In fact, the map $z \mapsto z^{\frac{1}{2}}$, or more generally $z \mapsto z^{\frac{1}{n}}$ with $n \geq 2$, is yet another example of a set-valued function on $\mathbf{C} \setminus \{0\}$. We'll have a lot more to say about such things in due course.

Time for a Break