Math 43: Spring 2020
Lecture 3 Part 2

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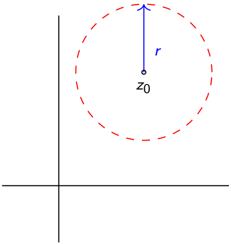
The Topology of $\mathbf{C}$

Remark

Since the complex numbers are really the plane $\mathbf{R}^2$ in disguise, we can "import" its structure from our multivariable calculus courses.

Definition

Let $z_0 \in \mathbf{C}$ and $r > 0$. Then $B_r(z_0) = \{ z \in \mathbf{C} : |z - z_0| < r \}$ is called the open ball of radius r centered at z_0.



Open and Closed Sets

Definition

Let $U \subset \mathbf{C}$ and $z_0 \in U$. We say that z_0 is an interior point of U if there is a $r > 0$ such that $B_r(z_0) \subset U$. We say that $U \subset \mathbf{C}$ is open if every point in U is an interior point. We say that $F \subset \mathbf{C}$ is closed if its complement $U := \mathbf{C} \setminus F$ is open.

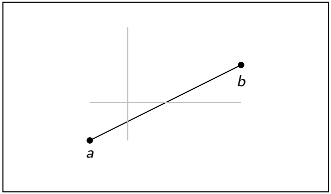
Example

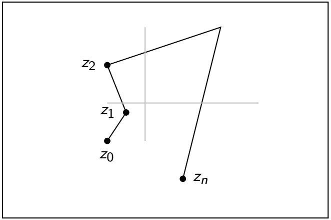
Consider the sets

$$U = \{\, z \in \mathbf{C} : 1 < \operatorname{Re} z < 2 \,\}$$
$$B = \{\, z \in \mathbf{C} : 1 \leq \operatorname{Re} z < 2 \,\}$$
$$F = \{\, z \in \mathbf{C} : 1 \leq \operatorname{Re} z \leq 2 \,\}.$$

Note that U is open, that F is closed, and that B is neither open nor closed.

If $a, b \in \mathbf{C}$, then the line segment from a to b is the set $[a, b] := \{ a + t(b - a) \in \mathbf{C} : t \in [0, 1] \}$. If $\{ z_0, z_1, \ldots, z_n \}$ are points in $\mathbf{C}$, then $\bigcup_{j=1}^{n}[z_{j-1}, z_j]$ is called a polygonal path from z_0 to z_n.

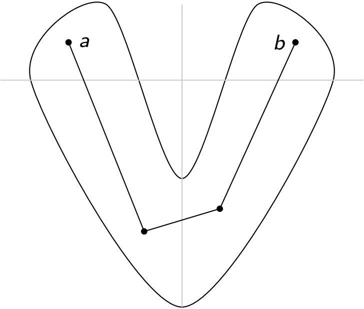




Connected Sets and Domains

Definition

An open set $D \subset \mathbf{C}$ is called connected if every pair of points a and b in D can be joined by polygonal path from a to b that lies entirely in D. A connected open subset of $\mathbf{C}$ is called a domain.

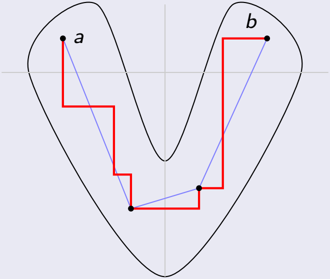


A Lemma

Lemma

If D is a domain, every pair of points in D can be joined by a polygonal path each line segment of which is parallel to one of the coordinate axes.

A Picture Proof.



□

A Little Multivariable Calculus

Theorem

Suppose that $D \subset \mathbf{C} = \mathbf{R}^2$ is a domain and that $u : D \subset \mathbf{C} \to \mathbf{R}$ is a real-valued function such that

$$\frac{\partial u}{\partial x}(a, b) = u_x(a, b) = 0 = u_y(a, b) = \frac{\partial u}{\partial y}(a, b)$$

for all $(a, b) \in D$. Then u is constant on D. Here we are writing (a, b) instead of $a + ib$ because this is really a result from multivariable calculus.

Remark

The key to the proof on the next slide is the observation that if $u_x \equiv 0$, then $x \mapsto u(x, y_0)$ must be constant for each fixed y_0. This is because $u_x(\cdot, y_0)$ is just the derivative of this function. Similarly, if $u_y \equiv 0$, then $y \mapsto u(x_0, y)$ is constant for each fixed x_0.

Proof of the Theorem

Proof of the Theorem on the Previous Slide.

Fix $(a,b) \in D$. It will suffice to see that for all $(x,y) \in D$ we have $u(x,y) = u(a,b)$. Since D is a domain, we can with the help of our unproved lemma, join (a,b) to (x,y) with a polygonal path $\bigcup_{j=1}^{n}[z_{j-1}, z_j]$ with each segment parallel to a coordinate axis. Furthermore, $z_0 = (a,b)$ and $z_n = (x,y)$. But the remark on the previous slide implies that u is constant on each segment. Thus

$$u(a,b) = u(z_0) = u(z_1) = \cdots = u(z_n) = u(x,y).$$

This is what we wanted to show. □

Time for a Break