Limits of Functions

Definition

Suppose that f is a complex-valued function defined on a deleted neighborhood of z_0. Then we say that $\lim_{z \to z_0} f(z) = w_0$ if for all $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |z - z_0| < \delta$ implies that $|f(z) - w_0| < \epsilon$.

Theorem

Suppose that $f(x + iy) = u(x, y) + iv(x, y)$ *is defined on a deleted neighborhood of* $z_0 = x_0 + iy_0$*. Then* $\lim_{z \to z_0} f(z) = a_0 + ib_0$ *if and only if*

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = a_0 \quad \textit{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = b_0.$$

Continuity

Definition

We say that $f : D \subset \mathbf{C} \to \mathbf{C}$ is continuous at $z_0 \in D$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Proposition

Suppose $f, g : D \subset \mathbf{C} \to \mathbf{C}$ are continuous at z_0. Then so are $f + g$ and fg. If $g(z_0) \neq 0$, then so is $\frac{f}{g}$. Moreover if h is continuous at $f(z_0)$, then $k(z) = h(f(z))$ is continuous at z_0.

Examples

Example

Part of the point of the lecture is that most of the functions we are familiar with—provided they still make sense—are continuous. Polynomials are everywhere continuous and rational functions are continuous wherever they make sense ("on their natural domains"). Our new complex exponential function is continuous. In fact, just about anything we can build using elementary algebra and composition are continuous on their natural domains. We just have to be careful not to write down things that don't make sense like $f(z) = \sqrt{e^z + z^2}$.

Complex Differentiation

Definition

Suppose f is defined in a neighborhood of z_0. Then we say that f is complex differentiable at z_0 if the complex limit

$$f'(z_0) = \lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w}$$

exists.

Back to Calculus I

Theorem

1. $\frac{d}{dz}\left(z^n\right) = nz^{n-1}$ *for all* $n \in \mathbf{Z}$.
2. $(f+g)' = f' + g'$.
3. $(fg)' = f'g + fg'$
4. $\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$.
5. *If* $h(z) = f(g(z))$, *then* $h'(z) = f'(g(z))g'(z)$.

Definition

Functions $f : \mathbf{C} \to \mathbf{C}$ that are complex differentiable everywhere are called entire functions.

Remark

All polynomials are entire functions. Rational functions are complex differentiable on their natural domains. So all the polynomial calculus from high school remains in place:
$\frac{d}{dz}\left(z + \frac{1}{z}\right)^{10} = 10\left(z + \frac{1}{z}\right)^9\left(1 - \frac{1}{z^2}\right)$.

Analytic Functions

Definition

Let $f : D \subset \mathbf{C} \to \mathbf{C}$ be a function. We say that f is analytic at $z_0 \in D$ of there is a neighborhood $U \subset D$ of z_0 such that $f'(z)$ exists for all $z \in U$. If D is a domain, then we say that f is analytic on D if $f'(z)$ exists for all $z \in D$.

Example

The function $f(z) = |z|^2$ is complex differentiable only at $z = 0$! Hence is not analytic at a single point.