Math 43: Spring 2020
Lecture 4 Part 1

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Limits of Sequences

Definition

A sequence $(z_n)_{n=1}^{\infty} \subset \mathbf{C}$ converges to a limit $z_0 \in \mathbf{C}$ if for all $\epsilon > 0$ there is a N such that $n \geq N$ implies $|z_n - z_0| < \epsilon$.

Remark

When (z_n) has a limit z_0 we often write $z_0 = \lim_{n\to\infty} z_n$ or sometimes just $z_n \to z_0$.

Theorem

Suppose that $z_n = x_n + iy_n$ and $z_0 = x_0 + iy_0$. Then $z_n \to z_0$ if and only if $x_n \to x_0$ and $y_n \to y_0$.

Proof.

This is just a restatement of a result from multivariable calculus. A sequence of vectors converges in $\mathbf{R}^n$ if and only if the components of the vectors converge. $\square$

Examples

Example

Let $z_n = \left(\frac{1+i}{\sqrt{3}-i}\right)^n$. Notice that $|z_n| = \left|\frac{1+i}{\sqrt{3}-i}\right|^n = \left(\frac{\sqrt{2}}{2}\right)^n \to 0$. But if $z_n = x_n + iy_n$, then $0 \leq |x_n| \leq \sqrt{x_n^2 + y_n^2} = |z_n|$. By the Squeeze Theorem, we must have $x_n \to 0$. Similarly, $y_n \to 0$. Therefore $z_n \to 0$!

Example

Now let $z_n = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^n$. Let $w = z_1$. Notice that $z_n = \left(e^{i\frac{2\pi}{3}}\right)^n = w^n$. Then $z_1, z_2, z_3, \dots = w, w^2, 1, w, w^2, \dots$. Therefore in this case, $\lim_{n\to\infty} z_n$ does not exist. (Think geometrically: z_n travels around the unit circle continuously without getting close to any particular point.)

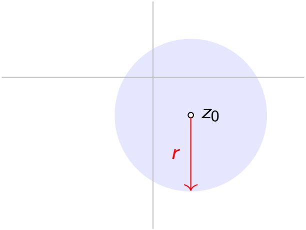
Deleted Neighborhoods

Definition

If $z_0 \in \mathbf{C}$ and $r > 0$, then

$$B_r'(z_0) = \{\, z \in \mathbf{C} : 0 < |z - z_0| < r \,\}$$

is called a deleted ball of radius r centered at z_0. It is also called a punctured disk of radius r centered at z_0 or even just a deleted neighborhood of z_0.



Limits of Functions

Definition

Suppose that f is a complex-valued function defined on a deleted neighborhood of z_0. Then we say that

$$\lim_{z \to z_0} f(z) = w_0$$

if for all $\epsilon > 0$ there is a $\delta > 0$ so that $0 < |z - z_0| < \delta$ implies that $|f(z) - w_0| < \epsilon$.

Remark

If we choose to view f as a vector valued function defined on a deleted neighborhood of the ordered pair z_0, then the limit exists as above exactly when the limit exists as a vector field as defined in our multivariable calculus courses.

Not to sweat the ϵ's and δ's

Theorem

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is defined on a deleted neighborhood of $z_0 = x_0 + iy_0$. Then

$$\lim_{z \to z_0} f(z) = a_0 + ib_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = a_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = b_0.$$

Proof.

This is a standard result from multivariable calculus provide with think of f as a function from $D \subset \mathbf{R}^2$ to $\mathbf{R}^2$. $\square$

Theorem

Suppose that $\lim_{z \to z_0} f(z) = w_0$ *and* $\lim_{z \to z_0} g(z) = w_1$. *Then*

1. $\lim_{z \to z_0} f(z) + g(z) = w_0 + w_1$,
2. $\lim_{z \to z_0} f(z)g(z) = w_0 w_1$, *and*
3. *if* $w_1 \neq 0$, *then* $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$.

Proof.

The proof is a tedious exercise in writing these formulas out in terms of the real and imaginary parts of f, g, w_0, and w_1. Then we can apply the corresponding results for real-valued functions of two variables. I'll shamelessly leave the details to you. □

Continuity

Definition

We say that $f : D \subset \mathbf{C} \to \mathbf{C}$ is continuous at $z_0 \in D$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Remark (NC-17 Warning)

This is the adult mathematician version of this definition. Back at Enormous State University, when I was teaching Business Calculus, the definition of continuity had three parts. One, f is defined in neighborhood of z_0 (and not just in a deleted neighborhood). Second, the limit $\lim_{z \to z_0} f(z)$ exists. Third, the limit equals $f(z_0)$.

Multivariable Calculus Again

Remark

Note that $f : D \subset \mathbf{C} \to \mathbf{C}$ is continuous if and only if $f : D \subset \mathbf{R}^2 \to \mathbf{R}^2$ is continuous as a vector field. The definitions are identical once we translate into vector calculus terms. This means that $f(x + iy) = u(x, y) + iv(x, y)$ is continuous at $z_0 = a + ib \in D$ exactly when both u and v are continuous at (a, b) as functions from $D \subset \mathbf{R}^2$ to $\mathbf{R}$.

Proposition

Suppose $f, g : D \subset \mathbf{C} \to \mathbf{C}$ are continuous at z_0. Then so are $f + g$ and fg. If $g(z_0) \neq 0$, then so is $\dfrac{f}{g}$. Moreover if h is continuous at $f(z_0)$, then $k(z) = h(f(z))$ is continuous at z_0.

Examples

1. It is not hard to check that constant functions and the identity function $f(z) = z$ are continuous everywhere. Hence a polynomial function $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is continuous everywhere for any constants $a_k \in \mathbf{C}$.
2. Hence a rational function $r(z) = \dfrac{p(z)}{q(z)}$ is continuous on its natural domain.
3. Since $f(x + iy) = \exp(x + iy) = e^x \cos(y) + ie^x \sin(y)$, we see that the complex exponential function $f(z) = e^z$ is continuous everywhere.
4. Let $f(z) = \dfrac{e^{z^2} - z}{z^3 + 1}$. Since the numerator and denominator are "clearly" continuous, the quotient is continuous on its natural domain. Thus f is continuous on $\mathbf{C} \setminus \{ -1, e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}} \} = \mathbf{C} \setminus \{ -1, \frac{1}{3} + i\frac{\sqrt{3}}{2}, \frac{1}{3} - i\frac{\sqrt{3}}{2} \}$.

Infinite Limits

Definition

Suppose that f is defined on a deleted neighborhood of z_0. Then we write $\lim_{z \to z_0} f(z) = \infty$ if $\lim_{z \to z_0} |f(z)| = \infty$.

Example

Find $\lim_{z \to 1+i} \dfrac{2z+1}{z^2-2i}$.

Solution.

Since the denominator vanishes at $1+i$, we can't just take the quotient of the limits. Instead notice that $\left|\dfrac{2z+1}{z^2-2i}\right| = \dfrac{|2z+1|}{|z^2-2i|}$. Since $\lim_{z \to 1+i} |2z+1| = \sqrt{13}$, while $\lim_{z \to 1+i} |z^2-2i| = 0$, the limit of their quotient tends to ∞. Hence $\lim_{z \to 1+i} \dfrac{2z+1}{z^2-2i} = \infty$. $\square$

Careful

Remark (**R** vs **C**)

In first year calculus, we learn that $\lim_{x\to 0} \frac{1}{x}$ does not exist because $\frac{1}{x}$ diverges to ∞ for small positive values of x while it diverges to $-\infty$ for negative values of x near 0. But since **C** isn't ordered, our definition says that $\lim_{z\to 0} \frac{1}{z} = \infty$!

Break Time