Math 43: Spring 2020
Lecture 4 Part 2

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Complex Differentiation

Now it is time for something completely new!

Definition

Suppose f is defined in a neighborhood of z_0. Then we say that f is complex differentiable at z_0 if the complex limit

$$f'(z_0) = \lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w}$$

exists.

Remark

While this definition looks exactly the same as our Calculus I definition, it is subtly different. Since the limit is taken in the field $\mathbf{C}$, it is a two-dimensional limit. The limit variable w can approach 0 in complicated ways!

Polynomial Calculus

Example

Let $f(z) = z^2$. Then

$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w} = \lim_{w \to 0} \frac{2zw + w^2}{w} = 2z.$$

Example

Consider $f(z) = |z|^2$.

Solution.

$$\begin{aligned} f'(z) &= \lim_{w \to 0} \frac{|z+w|^2 - |z|^2}{w} = \lim_{w \to 0} \frac{(z+w)(\overline{z} + \overline{w}) - z\overline{z}}{w} \\ &= \lim_{w \to 0} \frac{w\overline{z} + \overline{w}z + w\overline{w}}{w} = \lim_{w \to 0} \overline{z} + \overline{w} + z\frac{\overline{w}}{w} \\ &= \overline{z} + z \lim_{w \to 0} \frac{\overline{w}}{w}. \end{aligned}$$

So When Does the Limit Exist?

Solution Continued.

Thus if $z = 0$, then $f'(0) = 0$. But if $z \neq 0$, then the issue boils down to asking when does $\lim_{w \to 0} \frac{\overline{w}}{w}$ exist! Note that if the limit exists then we can approach 0 however we like and the value must always be the same. Thus we could take w of the form $h + i0$ with $h \in \mathbf{R}$. then $\lim_{w \to 0} \frac{\overline{w}}{w} = \lim_{h \to 0} \frac{h}{h} = 1$. On the other hand, we could let w be of the form $0 + ik$ with $k \in \mathbf{R}$. Then $\lim_{w \to 0} \frac{\overline{w}}{w} = \lim_{k \to 0} \frac{\overline{ik}}{ik} = -1$. We have shown that $f'(z)$ exists only when $z = 0$. Otherwise, f is not differentiable. This despite the fact that $f(x + iy) = x^2 + y^2 + i0$ is about as nice a function of x and y as you can imagine. $\square$

Remark

Whether you choose to let this example fill you with fear and trepidation or not, it does show that complex differentiation is a good deal more interesting that its real counterpart.

Back to Calculus I

Theorem

1. $\frac{d}{dz}\big(z^n\big) = nz^{n-1}$ *for all* $n \in \mathbf{Z}$.
2. $(f+g)' = f' + g'$.
3. $(fg)' = f'g + fg'$
4. $\big(\frac{f}{g}\big) = \frac{f'g - fg'}{g^2}$.
5. *If* $h(z) = f(g(z))$, *then* $h'(z) = f'(g(z))g'(z)$.

Definition

Functions $f : \mathbf{C} \to \mathbf{C}$ that are complex differentiable everywhere are called entire functions.

Remark

All polynomials are entire functions. Rational functions are complex differentiable on their natural domains. So all the polynomial calculus from high school remains in place:
$\frac{d}{dz}\big(z + \frac{1}{z}\big)^{10} = 10\big(z + \frac{1}{z}\big)^9\big(1 - \frac{1}{z^2}\big)$.

Neighborhoods

Definition

An open set U containing z_0 is called a neighborhood of z_0.

Remark

Notice that U is a neighborhood of z_0 if and only if there is a $r > 0$ such that $B_r(z_0) \subset U$. In particular, $B_r(z_0)$ is always a neighborhood of z_0 (provided $r > 0$).

Definition

Let $f : D \subset \mathbf{C} \to \mathbf{C}$ be a function. We say that f is analytic at $z_0 \in D$ of there is a neighborhood $U \subset D$ of z_0 such that $f'(z)$ exists for all $z \in U$. If D is a domain, then we say that f is analytic on D if $f'(z)$ exists for all $z \in D$.

Technicalities

Example

Let

$$f(z) = \frac{z^4}{z^2 + 1}.$$

Then f is analytic on $D = \mathbf{C} \setminus \{ \pm i \}$.

Example

Let $f(z) = |z|^2$. Then f is not analytic at a single point. However, f is complex differentiable at 0.

That is enough for now!