The Cauchy-Riemann Equations

Theorem (Cauchy-Riemann I)

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ *is complex differentiable at* $z_0 = x_0 + iy_0$. *Then*

$$f'(z_0) = f_x(z_0) = -if_y(z_0).$$

In particular, both u *and* v *have first partials at* (x_0, y_0) *and*

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0). \tag{1}$$

Remark

We call (1) the Cauchy-Riemann Equations for f at $z_0 = x_0 + iy_0$.

Remark (Obvious Question)

If the Cauchy-Riemann equations hold at z_0, does it follow that $f'(z_0)$ exists? The answer, unfortunately, is "no". A complicated example is given in problem #4 in Section 2.4 of the text. This means that the converse of Cauchy-Riemann Theorem I is false. Fortunately, the converse is "almost" true. But we had to work very hard to prove this.

The Converse

Theorem (Cauchy-Riemann II)

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is defined on $D = B_r(z_0)$ for some $r > 0$, and that that the Cauchy-Riemann equations for f are satisfied at $z_0 = x_0 + iy_0$. Suppose in addition that

1. *u and v have first partials in all of D, and that*
2. *these partials are continuous at (x_0, y_0).*

Then f is complex differentiable at z_0.

The Payoff

Corollary

Suppose that $D \subset \mathbf{C}$ *is a domain and* $f : D \subset \mathbf{C} \to \mathbf{C}$ *is given by* $f(z) = u(z) + iv(z)$. *If* u *and* v *both have continuous first partials in* D *and satisfy the Cauchy-Riemann equations at every point of* D, *then* f *is analytic in* D.

Corollary

Let $f(z) = e^z$. *Then* f *is entire and* $f'(z) = e^z$ *for all* $z \in \mathbf{C}$.

Zero Derivative

Theorem

Suppose that f is analytic on a domain D and that $f'(z) = 0$ for all $z \in D$. Then f is constant on D.

Theorem

Suppose that f is analytic on a domain D. Suppose also that $f(z) \in \mathbf{R}$ for all $z \in D$. Then f is constant.

Remark

In the homework for this lecture, you will discover that other quite reasonable restrictions on analytic functions on a domain force the function to be constant.