Math 43: Spring 2020
Lecture 5 Part 1

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Taking Complex Derivatives

Let $f(x+iy) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$ and $w = h + ik$. Suppose that $f'(z_0)$ exists. Then

$$\begin{aligned}
f'(z_0) &= \lim_{w\to 0} \frac{f(z_0+w) - f(z_0)}{w} \\
&= \lim_{(h,k)\mapsto(0,0)} \frac{u(x_0+h, y_0+k) + iv(x_0+h, y_0+k) - u(x_0,y_0) - iv(x_0,y_0)}{h+ik} \\
&= \lim_{h\to 0}\Bigl[\frac{u(x_0+h,y_0) - u(x_0,y_0)}{h} + i\frac{v(x_0+h,y_0) - v(x_0,y_0)}{h}\Bigr] \\
&= u_x(x_0,y_0) + iv_x(x_0,y_0) \\
&= f_x(x_0+iy_0) = f_x(z_0).
\end{aligned}$$

Remark

Cool. If $f'(z_0)$ exists, then $f'(z_0) = f_x(z_0) = u_x(z_0) + iv_x(z_0)$!

Wait a Minute!

But if $f'(z_0)$ exists, then we must also have

$$\begin{aligned}
f'(z_0) &= \lim_{w\to 0} \frac{f(z_0+w)-f(z_0)}{w} \\
&= \lim_{(h,k)\mapsto(0,0)} \frac{u(x_0+h,y_0+k)+iv(x_0+h,y_0+k)-u(x_0,y_0)-iv(x_0,y_0)}{h+ik} \\
&= \lim_{k\to 0}\Big[\frac{u(x_0,y_0+k)-u(x_0,y_0)}{ik} + i\frac{v(x_0,y_0+k)-v(x_0,y_0)}{ik}\Big] \\
&= -iu_y(x_0,y_0)+v_y(x_0,y_0) \\
&= -if_y(x_0+iy_0) = -if_y(z_0)!
\end{aligned}$$

Remark

If $f'(z_0)$ exists, then we also have $f'(z_0) = -if_y(z_0)$.

The Cauchy-Riemann Equations

Theorem (Cauchy-Riemann I)

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ *is complex differentiable at* $z_0 = x_0 + iy_0$. *Then*

$$f'(z_0) = f_x(z_0) = -if_y(z_0).$$

In particular, both u *and* v *have first partials at* (x_0, y_0) *and*

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \textit{and} \quad u_y(x_0,y_0) = -v_x(x_0,y_0). \qquad (1)$$

Remark

We call (2) the Cauchy-Riemann Equations for f at $z_0 = x_0 + iy_0$.

Complex Conjugation

Example

Consider the function $f(z) = \bar{z}$. That is, $f(x + iy) = x - iy$. Hence $u(x, y) = x$ and $v(x, y) = -y$. Then $u_x \equiv 1$ while $v_y \equiv -1$. Hence u_x is never equal to v_y. Hence the Cauchy-Riemann equations never hold. Therefore $f(z) = \bar{z}$ is not complex differentiable at a single point!.

Remark (Obvious Question)

If the Cauchy-Riemann equations hold at z_0, does it follow that $f'(z_0)$ exists? The answer, unfortunately, is "no". A complicated example is given in problem #4 in Section 2.4 of the text. This means that the converse of Cauchy-Riemann Theorem I is false. Fortunately, the converse is "almost" true. But we will have to work very hard to prove this.

The Converse

Theorem (Cauchy-Riemann II)

Suppose that $f(x+iy) = u(x,y) + iv(x,y)$ is defined on $D = B_r(z_0)$ for some $r > 0$, and that that the Cauchy-Riemann equations for f are satisfied at $z_0 = x_0 + iy_0$. Suppose in addition that

1. *u and v have first partials in all of D, and that*
2. *these partials are continuous at (x_0, y_0).*

Then f is complex differentiable at z_0.

Remark

The proof is quite involved. But I think the result is fundamental enough that it justifies the pain of working through it in detail. You may want to bring up the accompanying slides in a separate window.

Back in the Day

We'll need some good old fashioned calculus.

Theorem (Mean Value Theorem)

Suppose that $\varphi : [c, d] \subset \mathbf{R} \to \mathbf{R}$ is continuous and differentiable on (c, d). Then there is a point $t^* \in (c, d)$ such that

$$\frac{\varphi(d) - \varphi(c)}{d - c} = \varphi'(t^*). \tag{2}$$

We will use this result in the following form.

Corollary

Suppose that $\varphi : (c, d) \to \mathbf{R}$ is differentiable. Then if $a, a + h \in (c, d)$,

$$\varphi(a + h) - \varphi(a) = \varphi'(a^*)h$$

for an a^* strictly between a and $a + h$. In particular, $a^* \to a$ as $h \to 0$.

The Proof

We need to prove that $\lim_{w\to 0} \frac{f(z_0+w)-f(z_0)}{w}$ exists. Let $w = h + ik$ and assume that h and k are small enough so that $z_0 + w \in D$. Then

$$\begin{aligned}
\frac{f(z_0+w)-f(z_0)}{w} \\
&= \frac{u(x_0+h, y_0+k) + iv(x_0+h, y_0+k) - u(x_0,y_0) - iv(x_0,y_0)}{h+ik} \\
&= \underbrace{\frac{u(x_0+h, y_0+k) - u(x_0,y_0)}{h+ik}}_{\text{real part}} + i \underbrace{\frac{v(x_0+h, y_0+k) - v(x_0,y_0)}{h+ik}}_{\text{imaginary part}}
\end{aligned}$$

Using our MVT Corollary, the numerator of the real part is

$$\begin{aligned}
u(x_0+h, y_0+k) - u(x_0, y_0+k) + u(x_0, y_0+k) - u(x_0,y_0) \\
&= u_x(x_0^*, y_0+k)h + u_y(x_0, y_0^*)k
\end{aligned}$$

where we know that $(x_0^*, y_0^*) \to (x_0, y_0)$ as $(h,k) \to (0,0)$.

Proof Continued

Now since u_x and u_y are continuous at (x_0, y_0), $u_x(x_0^*, y_0 + k) = u_x(x_0, y_0) + \epsilon_1(h, k)$ where $\epsilon_1(h, k) \to 0$ and $(h, k) \to (0, 0)$. Similarly, $u_y(x_0, y_0^*) = u_y(x_0, y_0) + \epsilon_2(h, k)$ and $\epsilon_2(h, k) \to 0$ and $(h, k) \to (0, 0)$. This means we can write the numerator of the real part as

$$(A) \qquad u_x(z_0)h + u_y(z_0)k + \epsilon_1(h, k)h + \epsilon_2(h, k)k.$$

Similarly, we can write the numerator of the imaginary part in the form

$$(B) \qquad v_x(z_0)h + v_y(z_0)k + \epsilon_3(h, k)h + \epsilon_4(h, k)k.$$

Then $\frac{A+iB}{h+ik}$ simplifies to

$$\frac{h(u_x(z_0) + iv_x(z_0)) + k(u_y(z_0) + iv_y(z_0)) + h(\epsilon_1 + i\epsilon_3) + k(\epsilon_2 + i\epsilon_4)}{h + ik}$$

Since the CR-eqns imply $k(u_y + iv_y) = ik(-iu_y + v_y) = ik(u_x + iv_x)$, the above can be written as

$$u_x(z_0) + iv_x(z_0) + \underbrace{\frac{h(\epsilon_1 + i\epsilon_3) + k(\epsilon_2 + i\epsilon_4)}{h + ik}}_{\text{mess}}.$$

Finish the Proof

Wow! Now we can finish the proof if we can show that the "mess" goes to zero as $w \to 0$. But

$$|\text{mess}| \le \left|\frac{h}{h+ik}\right| |\epsilon_1 + i\epsilon_3| + \left|\frac{k}{h+ik}\right| |\epsilon_2 + i\epsilon_4|$$

$$\le |\epsilon_1 + i\epsilon_3| + |\epsilon_2 + i\epsilon_4|$$

which tends to 0 as $(h,k) \to (0,0)$. Thus

$$\lim_{z \to 0} \frac{f(z_0 + w) - f(z_0)}{w} = u_x(z_0) + iv_x(z_0)$$

and we're done.

Now it is definitely time for a break.