Math 43: Spring 2020
Lecture 5 Part 2

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The Payoff

Now that we have a sufficient condition for complex differentiability, we we can prove the following.

Corollary

Suppose that $D \subset \mathbf{C}$ is a domain and $f : D \subset \mathbf{C} \to \mathbf{C}$ is given by $f(z) = u(z) + iv(z)$. If u and v both have continuous first partials in D and satisfy the Cauchy-Riemann equations at every point of D, then f is analytic in D.

Proof.

Since D is open, it suffices to see that $f'(z)$ exists at each point of D. But this follows from the Cauchy-Riemann Theorem II!. $\square$

The Complex Exponential Function

Corollary

Let $f(z) = e^z$. Then f is entire and $f'(z) = e^z$ for all $z \in \mathbf{C}$.

Proof.

We have $f(x+iy) = e^x \cos(y) + ie^x \sin(y)$. Then $u(x,y) = e^x \cos(y)$ and $v(x,y) = e^x \sin(y)$. We easily see that the first partials are continuous and that

$$u_x(z) = v_y(z) \quad \text{and} \quad u_y(z) = -v_x(z)$$

for all z. Hence f is differentiable at all z by our Cauchy-Riemann Theorem II. But by our Cauchy-Riemann Theorem I,

$$f'(x+iy) = u_x(x,y) + iv_x(x,y) = e^x \cos(y) + ie^x \sin(y) = f(x+iy).$$

$\square$

I Told You So!

Remark

Our motivation for the definition of the complex exponential function $f(z) = e^z$ was a bit *ad hoc*. Our first "excuse" for this definition, $e^{x+iy} := e^x\big(\cos(y) + i\sin(y)\big)$, was that $z \mapsto e^z$ had the nice "exponent properties" that $e^{z+w} = e^z e^w$, $e^{-z} = \frac{1}{e^z}$, etc. But now we have the satisfaction of knowing that $\frac{d}{dz}e^z = e^z$. We will gather more certainty that our definition is the "right one" as we go further in the course.

Zero Derivative

Theorem

Suppose that f is analytic on a domain D and that $f'(z) = 0$ for all $z \in D$. Then f is constant on D.

Proof.

By our first CR theorem,

$$f'(z) = f_x(z) = u_x(z) + iv_x(z) = -if_y(z) = v_y(z) - iu_y(z).$$

Hence if $f'(z) \equiv 0$, then $u_x \equiv 0 \equiv u_y$. We proved that this implies u is constant. Similarly, $v_x \equiv 0 \equiv v_y$ and v is constant. Thus f is constant. $\square$

Example

Example

Show that f is an entire function and $f'(z) = f(z)$ for all z. Show that $f(z) = ae^z$ for some $a \in \mathbf{C}$.

Solution.

Since e^z never vanishes, the function $h(z) = \frac{f(z)}{e^z}$ is entire. But

$$h'(z) = \frac{f'(z)e^z - f(z)\frac{d}{dz}e^z}{(e^z)^2} = \frac{f(z)e^z - f(z)e^z}{e^{2z}} = 0 \quad \text{for all } z \in \mathbf{C}.$$

It follows from the previous theorem that h is constant. Thus there is a $a \in \mathbf{C}$ such that $h(z) = a$ for all $z \in \mathbf{C}$. □

Theorem

Suppose that f is analytic on a domain D. Suppose also that $f(z) \in \mathbf{R}$ for all $z \in D$. Then f is constant.

Proof.

We have $f(z) = u(z) + iv(z)$ with $v \equiv 0$. Thus $v_x \equiv 0 \equiv v_y$. Then by CR Thm I,

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) + iv_x(z) = 0 \quad \text{for all } z \in D.$$

Hence f is constant. □

Remark

In the homework for this lecture, you will discover that other quite reasonable restrictions on analytic functions on a domain force the function to be constant.

That is enough for now!