Harmonic Functions

Definition

Let $u : D \subset \mathbf{R}^2 \to \mathbf{R}$ have continuous second partials in D. We say that u is harmonic in D if

$$\frac{\partial^2 u}{\partial x^2}(a,b) + \frac{\partial^2 u}{\partial y^2}(a,b) = 0 = u_{xx}(a,b) + u_{yy}(a,b)$$

for all $(a,b) \in D$.

Example

The function $u(x,y) = e^x \cos(y)$ is harmonic on all of $\mathbf{R}^2$.

Theorem

Suppose that $f(z) = u(z) + iv(z)$ is analytic in a domain D. Suppose also that u and v have continuous second partials in D. Then u and v are harmonic in D.

Harmonic Conjugates

Definition

Suppose that u is harmonic on a domain D. We call v a harmonic conjugate for u on D if $f(z) = u(z) + iv(z)$ is analytic on D.

Example

Since $e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$ is entire, $v(x, y) = e^x \sin(y)$ is a harmonic conjugate for $u(x, y) = e^x \cos(y)$ in the complex plane. Of course, so is $v_1(x, y) = e^x \sin(y) + \sqrt{17}$.

Proposition

Suppose that v and w are both harmonic conjugates for u in a domain D. Then there is a real constant k such that

$$v(z) = w(z) + k \quad \text{for all } z \in D.$$

A Challenging Homework Problem

Remark

The question of whether or not a given harmonic function in a domain D always has a harmonic conjugate in D is very subtle. In general, it depends on the domain D. For homework this week, you are going to see that while $u(z) = \ln(|z|)$ is harmonic on the punctured plane $\mathbf{C} \setminus \{0\}$, it can not have a harmonic conjugate on the entire punctured plane! I provided some hints on the assignment page.

Back to High School

Definition

A complex polynomial is a function of the form $p(z) = a_0 + a_1 z + \cdots a_n z^n$ with each $a_k \in \mathbf{C}$ and $n \geq 0$. We will write $\mathbf{C}[z]$ for the complex vector space of all polynomial functions. If $a_n \neq 0$, then we call n the degree of $p(z)$ and write $\deg p(z) = n$. For technical reasons, we define the degree of the zero polynomial to be -1.

Theorem (Division Algorithm—aka Long Division)

Suppose that $p(z)$ and $q(z)$ are nonzero polynomials. Then there are unique polynomials $r(z)$ and $s(z)$ such that $p(z) = s(z)q(z) + r(z)$ such that $\deg r(z) < \deg q(z)$. Alternatively,

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)}$$

with $\deg r(z) < \deg q(z)$.

Roots

Definition

If $p(z) \in \mathbf{C}[z]$ and $\deg p(z) \geq 1$, then we call a a root of $p(z)$ if $p(a) = 0$.

Theorem

Suppose $p(z) \in \mathbf{C}[z]$ with $\deg p(z) \geq 1$. Then a is a root of $p(z)$ if and only if $p(z) = q(z)(z-a)$ with $q(z) \in \mathbf{C}[z]$. In this case, we say that $(z-a)$ divides $p(z)$ or that $(z-a)$ is a factor of $p(z)$.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial in $\mathbf{C}[z]$ with $\deg p(z) \geq 1$ has at least one root.

Corollary

Suppose that $p(z) \in \mathbf{C}[z]$ with $\deg p(z) = n \geq 1$. Then there are complex constants a and $z_1, \dots, z_n$ such that

$$p(z) = a(z-z_1)(z-z_2)\cdots(z-z_n).$$

What about over the Reals?

Definition (Canonical Bad Apple)

A quadratic polynomial $q(z) = az^2 + bz = c$ with real coefficients is called an irreducible quadratic if $q(z)$ has no real roots.

Example

$q(z) = z^2 + 1$ and $s(z) = z^2 + z + 1$ are examples of irreducible quadratics.

Theorem

Suppose that $p(z) \in \mathbf{C}[z]$ has all real coefficients. The $p(z)$ factors into a product of irreducible quadratics and real linear factors: $p(z) = a(z - r_1) \cdots (z - r_k) q_1(z) \cdots q_s(z)$ *with* $\deg p(z) = k + 2s$.