Math 43: Spring 2020
Lecture 6 Part 1

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Harmonic Functions

Definition

Let $u : D \subset \mathbf{R}^2 \to \mathbf{R}$ have continuous second partials in D. We say that u is harmonic in D if

$$\frac{\partial^2 u}{\partial x^2}(a,b) + \frac{\partial^2 u}{\partial y^2}(a,b) = 0 \quad \text{for all } (a,b) \in D.$$

Remark (Notation)

Just as will our notations for first partials in our discussion of the Cauchy-Riemann equations, we won't usually be so formal with our second partials. Instead we generally write $u_{xx} + u_{yy} = 0$.

Example

The function $u(x,y) = e^x \cos(y)$ is harmonic on all of $\mathbf{R}^2$.

Playing with Second Partials

Theorem (Clairaut's Theorem)

Suppose that $u : D \subset \mathbf{R}^2 \to \mathbf{R}$ *has continuous second partials on a Domain* D*. Then*

$$u_{xy}(a, b) = \frac{\partial^2 u}{\partial y \partial x}(a, b) = \frac{\partial^2 u}{\partial x \partial y}(a, b) = u_{yx}(a, b)$$

for all $(a, b) \in D$.

Remark

One nice consequence of Clairaut's Theorem is that we don't have to pay much attention to the subtle ordering of the x's and y's is the above notations.

Why Mention Harmonic Functions Here?

Remark

Harmonic functions have many important applications in both physics and engineering as well as being attractive mathematical objects. But the excuse for introducing them in an elementary course on complex analysis is the following.

Theorem

Suppose that $f(z) = u(z) + iv(z)$ is analytic in a domain D. Suppose also that u and v have continuous second partials in D. Then u and v are harmonic in D.

Remark

Later in the term, we will prove that the real and imaginary parts of an analytic function always have continuous partials of all orders. Hence the hypothesis in blue above turns out to be unnecessary.

The Proof

Proof.

We invoke the CR equations:

$$u_{xx} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y}$$

which, but Clairaut's Theorem, is

$$\begin{aligned} &= \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\ &= \frac{\partial}{\partial y}\Big(-\frac{\partial u}{\partial y}\Big) = -u_{yy}. \end{aligned}$$

$\square$

This shows that $u = \operatorname{Re} f$ is harmonic. Since $v = \operatorname{Re}(-if)$, v is also harmonic.

An Example

Example

It is not hard to check that $u(x,y) = y - 2xy$ is harmonic. Can we find an entire function f such that $u = \operatorname{Re} f$?

Solution.

If $f(z) = u(z) + iv(z)$ is such a function, then $v_x(x,y) = -u_y(x,y) = -(1-2x) = -1 + 2x$. It follows that $v(x,y) = -x + x^2 + c(y)$ where $c(y)$ depends only on y. But then $v_y = 0 + c'(y) = u_x(x,y) = -2y$. Hence $c(y) = -y^2 + k$ where $k \in \mathbf{R}$. This implies that $v(x,y) = -x + x^2 - y^2 + k$. By construction, the function $f(x+iy) = (y - 2xy) + i(x^2 - y^2 - x + k)$ satisfies the CR equations and and has continuous first partials. Hence it is entire by our CR Theorem II. Therefore we can find such a function in this case. Notice that $f(z) = iz^2 - iz + ik$. $\square$

Harmonic Conjugates

Definition

Suppose that u is harmonic on a domain D. We call v a harmonic conjugate for u on D if $f(z) = u(z) + iv(z)$ is analytic on D.

Example

$v(x, y) = x^2 - y^2 - x$ is a harmonic conjugate for $u(x, y) = y - 2xy$ on $\mathbf{C}$.

Uniqueness

Proposition

Suppose that v and w are both harmonic conjugates for u in a domain D. Then there is a real constant k such that

$$v(z) = w(z) + k \quad \textit{for all } z \in D.$$

Proof.

By assumption, $f(z) = u(z) + iv(z)$ and $g(z) = u(z) + iw(z)$ are analytic in D. But then so is

$$h(z) = -i\big(f(z) - g(z)\big) = -i\big(iv(z) - iw(z)\big) = v(z) - w(z).$$

Notice that h is real-valued in D. Therefore h is constant and there is a $k \in \mathbf{R}$ (since h is real-valued) such that

$$k = -i\big(f(z) - g(z)\big) = v(z) - w(z). \qquad \square$$

Getting Interesting

For the purposes of this lecture's homework assignment, you may assume the following result. Those interested can try to give a proof. I'll supply one in the solutions.

Proposition

1. *The function* $u(x, y) = \ln(|x + iy|) = \frac{1}{2}\ln(x^2 + y^2)$ *is harmonic on the punctured complex plane* $\mathbf{C} \setminus \{0\}$.
2. *The function* $v(x, y) = \operatorname{Arg}(x + iy)$ *is harmonic on* $D^* = \mathbf{C} \setminus (-\infty, 0]$.
3. *The function* $f(x + iy) = u(x, y) + iv(x, y) = \ln(|z + iy|) + i\operatorname{Arg}(x + iy)$ *is analytic in* D^*.

Sketch of the Proof.

Proving (1) is a computation. We can turn (2) into a computation by realizing $\operatorname{Arg}(z + iy)$ using inverse trig functions. Then (3) follows by our CR Theorem II. $\square$

A Challenging Homework Problem

For homework (problem #21 in section 2.5), you are asked to prove that $u(z) = \ln(|z|)$ does not have a harmonic conjugate on the entire punctured complex plane $\mathbf{C} \setminus \{0\}$. You are allowed to use the assertions in the previous proposition without proving them. This the first of many situations where the topology of the domain our functions are defined on comes into play. Here the "topology" in question amounts to the observation that $\mathbf{C} \setminus \{0\}$ has a hole in it!

Break Time!