Math 43: Spring 2020
Lecture 6 Part 2

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Back to High School

Most of the material in this part of the lecture is taken from §3.1 of the text.

Definition

A complex polynomial is a function of the form $p(z) = a_0 + a_1 z + \cdots a_n z^n$ with each $a_k \in \mathbf{C}$ and $n \geq 0$. We will write $\mathbf{C}[z]$ for the complex vector space of all polynomial functions. If $a_n \neq 0$, then we call n the degree of $p(z)$ and write $\deg p(z) = n$. For technical reasons, we define the degree of the zero polynomial to be -1.

The Division Algorithm

Theorem

Suppose that $p(z)$ and $q(z)$ are nonzero polynomials. Then there are unique polynomials $r(z)$ and $s(z)$ such that

$$p(z) = s(z)q(z) + r(z)$$

such that

$$\deg r(z) < \deg q(z).$$

Remark

Alternatively,

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)}$$

with $\deg r(z) < \deg q(z)$.

Example (Using the Document Camera)

Let $p(z) = z^3 + z^2 + 1$ and $q(z) = z^2 + 1$. Then

$$z^3 + z^2 + 1 = (z+1)(z^2+1) - x.$$

Alternatively,

$$\frac{z^3 + z^2 + 1}{z^2 + 1} = z + 1 - \frac{z}{z^2 + 1}.$$

Roots

Definition

If $p(z) \in \mathbf{C}[z]$ and $\deg p(z) \geq 1$, then we call a a root of $p(z)$ if $p(a) = 0$.

Theorem

Suppose $p(z) \in \mathbf{C}[z]$ with $\deg p(z) \geq 1$. Then a is a root of $p(z)$ if and only if $p(z) = q(z)(z - a)$ with $q(z) \in \mathbf{C}[z]$. In this case, we say that $(z - a)$ divides $p(z)$ or that $(z - a)$ is a factor of $p(z)$.

The Fundamental Theorem of Algebra

Theorem (The Fundamental Theorem of Algebra)

Every polynomial in $\mathbf{C}[z]$ *with* $\deg p(z) \geq 1$ *has at least one root.*

Remark

Originally Gauss was given credit for this result in 1799 after several incomplete proofs were put forward by other mathematicians. But even Gauss's proof is considered flawed. Today Argand is given credit for the first correct proof in 1806. We will give a pretty proof of this result later in this course!

Easy Consequence

Corollary

Suppose that $p(z) \in \mathbf{C}[z]$ *with* $\deg p(z) = n \geq 1$. *Then there are complex constants* a *and* $z_1, \dots, z_n$ *such that*

$$p(z) = a(z - z_1)(z - z_2) \cdots (z - z_n).$$

Remark

This result is really what makes the complex number field special. Thus "just adding i to the reals" not only gives us n^{th}-roots of all nonzero complex numbers but allows all polynomials to factor completely!

What about over the Reals?

Example

The polynomial $q(z) = z^6 + 1$ has real coefficients, but no real roots at all. So it has no linear factors whatsoever over $\mathbf{R}$!

Definition (Canonical Bad Apple)

A quadratic polynomial $q(z) = az^2 + bz = c$ with real coefficients is called an irreducible quadratic if $q(z)$ has no real roots.

Example

$q(z) = z^2 + 1$ and $s(z) = z^2 + z + 1$ are examples of irreducible quadratics.

Real Coefficients

Theorem

Suppose that $p(z) \in \mathbf{C}[z]$ *has all real coefficients. The* $p(z)$ *factors into a product of irreducible quadratics and real linear factors:* $p(z) = a(z - r_1) \cdots (z - r_k) q_1(z) \cdots q_s(z)$ *with* $\deg p(z) = k + 2s$.

Proof.

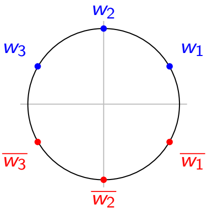
By the FTA, $p(z) = a(z - z_1) \cdots (z - z_n)$ with $a \in \mathbf{R}$ and $z_k \in \mathbf{C}$. But since $p(z)$ has real coefficients, $\overline{p(z)} = p(\overline{z})$. Hence if $p(z_k) = 0$, we also have $p(\overline{z_k}) = 0$. Thus either z_k is real, or $\overline{z_k}$ is also a root. Thus we can reorder so that $\{ z_1, \ldots, z_n \} = \{ r_1, \ldots, r_k, z_1, \overline{z_1}, \ldots, z_s, \overline{z_s} \}$ with each r_j real. But $(z - z_j)(z - \overline{z_j}) = z^2 - (z_j + \overline{z_j})z + |z_j|^2 = z^2 - 2\,\mathrm{Re}(z_j)z + |z_j|^2 := q_j(z)$. Consequently, $p(z) = a(z - r_1) \cdots (z - r_k) q_1(z) \cdots q_s(z)$. $\square$

Example

Example

Factor $q(z) = z^6 + 1$ over the reals.

The roots are the 6^{th}-roots of -1. Then $w_1 = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$, $w_2 = e^{i\frac{\pi}{2}} = i$, and $w_3 = e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{3}$. The remaining three



must be the complex conjugates.

Thus invoking the proof of the previous result,

$$\boxed{q(z) = (z^2 - \sqrt{3}z + 1)(z^2 + 1)(z^2 + \sqrt{3}z + 1)}.$$

Time to Rest

That's enough.