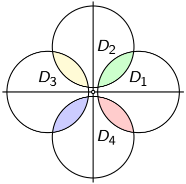
Harmonic Conjugates



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Well, u has a harmonic conjugate in disks D_1 and D_2. They must differ by a constant on the green overlap, so we can adjust the conjugate in D_2 to get a harmonic conjugate in the union $D_1 \cup D_2$. Then we can adjust the harmonic conjugate in D_3 to get a harmonic conjugate in $D_1 \cup D_2 \cup D_3$. But we have a problem when we try to adjust the harmonic conjugate in D_4. It has to match up in the blue overlap and the red overlap, and we may not be able to do that.

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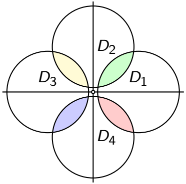

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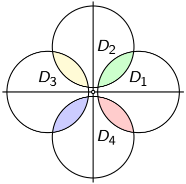

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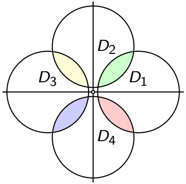

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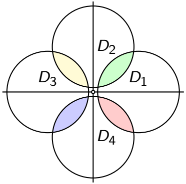

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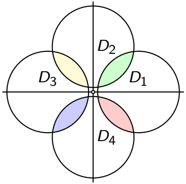

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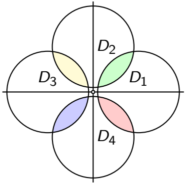

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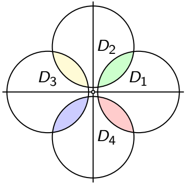

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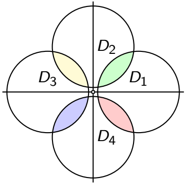

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Partial Fraction Decompositions over $\mathbf{C}$

Theorem

Suppose that

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{a(z-w_1)^{d_1}(z-w_2)^{d_2}\cdots(z-w_s)^{d_s}}$$

is a rational function with $\deg p(z) < \deg q(z) = d_1 + d_2 + \cdots + d_s$. *Then* $f(z) = r_1(z) + r_2(z) + \cdots + r_s(z)$ *where*

$$r_k(z) = \frac{A_0^{(k)}}{(z-w_k)^{d_k}} + \frac{A_1^{(k)}}{(z-w_k)^{d_k-1}} + \cdots + \frac{A_{d_k-1}^{(k)}}{(z-w_k)}$$

Not so Bad in Practice

Example

The previous result tells us that

$$f(z) = \frac{1}{z(z-1)^3} = \frac{A}{z} + \frac{B}{(z-1)^3} + \frac{C}{(z-1)^2} + \frac{D}{z-1}.$$

We discussed how we easily arrive at

$$\frac{1}{z(z-1)^3} = -\frac{1}{z} + \frac{B}{(z-1)^3} - \frac{1}{(z-1)^2} + \frac{1}{z-1}.$$

You can even memorize Eq (21) in Section 3.1 which gives a general formula for the $A_j^{(k)}$. But I think it easier to figure it our on a case by case basis. In particular, you can compute D in our example as

$$D = \lim_{z \to 1} \frac{1}{2} \frac{d^2}{dz^2}\big((z-1)^3 f(z)\big).$$

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The Complex Exponential Function

Theorem

1. *We have* $e^w = 1$ *if and only if* $w = 2\pi ik$ *for some* $k \in \mathbf{Z}$.
2. *We have* $e^z = e^w$ *if and only if* $w = z + 2\pi ik$ *for some* $k \in \mathbf{Z}$.

Corollary

The complex exponential function $f(z) = e^z$ *is periodic with period* $2\pi i$. *That is,* $f(z + 2\pi i) = f(z)$ *for all* $z \in \mathbf{C}$.

Some Other Entire Functions

Definition

For all $z \in \mathbf{C}$, we define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Theorem

The functions $f(z) = \cos(z)$ *and* $g(z) = \sin(z)$ *are entire with*

$$\frac{d}{dz}(\cos(z)) = -\sin(z) \quad \textit{and} \quad \frac{d}{dz}(\sin(z)) = \cos(z)$$

Further both are periodic with period 2π*.*