# Math 43: Spring 2020
# Lecture 7 Part 1

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# Partial Fraction Decompositions over $\mathbf{C}$

**Remark**

Using the Division Algorithm, analyzing rational functions reduces to considering the case where the degree of the numerator is strictly less than the degree of the denominator.

**Theorem**

*Suppose that*

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{a(z-w_1)^{d_1}(z-w_2)^{d_2}\cdots(z-w_s)^{d_s}}$$

*is a rational function with* $\deg p(z) < \deg q(z) = d_1 + d_2 + \cdots + d_s$. *Then* $f(z) = r_1(z) + r_2(z) + \cdots + r_s(z)$ *where*

$$r_k(z) = \frac{A_0^{(k)}}{(z-w_k)^{d_k}} + \frac{A_1^{(k)}}{(z-w_k)^{d_k-1}} + \cdots + \frac{A_{d_k-1}^{(k)}}{(z-w_k)}$$

# Not to Worry

## Remark

Because we need a notation that allows for the general case, the notation in the previous result is way more scary that it is in practice.

## Example

Let $f(z) = \dfrac{4z+4}{z(z-1)(z-2)^2}$. Then our partial fraction decomposition is supposed to be of the form

$f(z) = \dfrac{A_0^{(1)}}{z} + \dfrac{A_0^{(2)}}{z-1} + \dfrac{A_0^{(3)}}{(z-2)^2} + \dfrac{A_1^{(3)}}{z-2}$. for constants $A_i^{(j)}$. But in a particular example like this, there is no need to keep the pedantic notation in the theorem. Instead, we can just assume

$$f(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-2)^2} + \frac{D}{z-2}$$

for constants $A$, $B$, $C$, and $D$.

# Solving for the Constants

## Example (Continued)

Since $f(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{(z-2)^2} + \frac{D}{z-2}$ we can solve for $A$ easily: $A = \lim_{z\to 0} zf(z) = \lim_{z\to 0}\Big[A + z\big(\frac{B}{z-1} + \frac{C}{(z-2)^2} + \frac{D}{z-2}\big)\Big]$. But this limit is easy to evaluate! $A = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{4z+4}{(z-1)(z-2)^2} = -1$. Similarly, $B = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1} \frac{4z+4}{z(z-2)^2} = \frac{8}{1} = 8$. Moreover, we even have $C = \lim_{z\to 2}(z-2)^2 f(z) = \lim_{z\to 2} \frac{4z+4}{z(z-1)} = \frac{12}{2} = 6$.
But how can we find $D$ without a ridiculous amount of algebra?

# Finding $D$

## Example (Continued)

Observe that $(z-2)^2 f(z) = C + D(z-2) + (z-2)^2(r_1(z) + r_2(x))$. Then $\frac{d}{dz}\big((z-2)^2 f(z)\big) = D + (z-2)(g(z))$ with $g$ some rational function which is continuous at 2. Therefore

$$\begin{aligned} D &= \lim_{z\to 2} \frac{d}{dz}\big((z-2)^2 f(z)\big) = \lim_{z\to 2} \frac{d}{dz}\Big(\frac{4z+4}{z(z-1)}\Big) \\ &= \lim_{z\to 2} \frac{4(z^2-z) - (4z+4)(2z-1)}{(z^2-z)^2} = \frac{4(2) - 12(3)}{2^2} \\ &= \frac{8-36}{4} = -7. \end{aligned}$$

Hence

$$\frac{4z+4}{z(z-1)(z-2)^2} = -\frac{1}{z} + \frac{8}{z-1} + \frac{6}{(z-2)^2} - \frac{7}{z-2}.$$

# Time for a Break

That's all I'll have to say about Section 3.1 in lecture. We'll get back to complex functions in the next part of this lecture.

A winter view of New Hampshire from our backyard a few winters ago.