Math 43: Spring 2020
Lecture 7 Part 2

Dana P. Williams

Dartmouth College

Monday April 13, 2020

Back to Complex Analysis!

- Let's review.
- $\left|e^{x+iy}\right| = |e^x| \cdot |e^{iy}| = e^x$.
- $\arg e^{x+iy} = \arg e^x e^{iy} = \{\, y + 2\pi k : k \in \mathbf{Z} \,\}$.
- e^z is never zero.
- $\frac{d}{dz} e^z = e^z$.

The Complex Exponential Function

Theorem

1. *We have $e^w = 1$ if and only if $w = 2\pi ik$ for some $k \in \mathbf{Z}$.*
2. *We have $e^z = e^w$ if and only if $w = z + 2\pi ik$ for some $k \in \mathbf{Z}$.*

Proof.

Since $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we clearly have $e^{2\pi ik} = 1$ for any $k \in \mathbf{Z}$. Conversely if $e^{x+iy} = 1$, then $e^x = |e^{x+iy}| = 1$ and $x = 0$. But if $e^{iy} = \cos(y) + i\sin(y) = 1$, then $\cos(y) = 1$ and $\sin(y) = 0$. The later implies that $y = k\pi$ for some $k \in \mathbf{Z}$. But $\cos(k\pi) = 1$ only if k is even. This proves (1).

But if $e^z = e^w$, then $e^{w-z} = 1$. Hence (2) now follows from (1). $\square$

The Exponential Function is Periodic

Corollary

The complex exponential function $f(z) = e^z$ is periodic with period $2\pi i$. That is, $f(z + 2\pi i) = f(z)$ for all $z \in \mathbf{C}$.

Remark

Let $S_n = \{\, z \in \mathbf{C} : (2n-1)\pi < \operatorname{Im}(z) \leq (2n+1)\pi \,\}$. Then $z \mapsto e^z$ is one-to-one from S_n onto $\mathbf{C} \setminus \{0\}$. The one-to-one part comes from part (2) of the previous theorem since one and only one element of $\{\, z + 2\pi ik : k \in \mathbf{Z} \,\}$ can be in any S_n. To see that the map is onto, we note that we have already commented that e^w can be any nonzero complex number. But some element of $\{\, w + 2\pi ik : k \in \mathbf{Z} \,\}$ must be in S_n.

Draw a Picture

Some Other Entire Functions

Since for $\theta \in \mathbf{R}$, we have

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

this serves to motivate the following definition which allows us to extend the cosine and sine functions to all $\mathbf{C}$ just as we did for the exponential function.

Definition

For all $z \in \mathbf{C}$, we define

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

New Old Friends

Theorem

The functions $f(z) = \cos(z)$ and $g(z) = \sin(z)$ are entire with $\frac{d}{dz}(\cos(z)) = -\sin(z)$ and $\frac{d}{dz}(\sin(z)) = \cos(z)$. Further both are periodic with period 2π.

Proof.

Since $z \mapsto e^z$ is entire, it follows easily that both f and g are also. If $h(z) = e^{iz}$, then it is immediate that $h(z + 2\pi) = h(z)$ for all z. Hence both cosine and sine are periodic with period 2π. Also

$$\begin{aligned}\frac{d}{dz}(\cos(z)) &= \frac{d}{dz}\Big(\frac{e^{iz} + e^{-iz}}{2}\Big) \\ &= \frac{ie^{iz} - ie^{iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin(z).\end{aligned}$$

The proof that the derivative of sine is cosine is similar. $\square$

Complex Trigonmetric Functions

Remark

Of course, if you wish, and we do, you can now build all the usual trigonometric friends from back in the day such as $\tan(z) = \frac{\sin(z)}{\cos(z)}$ and so on. For the engineers, we also have the hyperbolic functions $\cosh(z) = \frac{e^z + e^{-z}}{2}$ and $\sinh(z) = \frac{e^z - e^{-z}}{2}$. Then if you wish, you can while away an hour or two checking that all your old common trigonometric identities still hold:

1. $\cos^2(z) + \sin^2(z) = 1$,
2. $\cos(2z) = \cos^2(z) - \sin^2(z)$,
3. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$,
4. etc.

Fortunately, we will eventually see that we can check that these identities hold without all the nasty algebra required at this point!

Not Your Father's Cosine

Remark (Question)

Since $\cos^2(z) + \sin^2(z) = 1$ just line in the old days, does it follow that

$$|\cos(z)| \leq 1 \quad \text{for all } z?$$

It turns out that the answer is emphatically NO. To see this immediately, observe that for $x \in \mathbf{R}$, we have

$$\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh(x) \geq 1.$$

Note that $\lim_{x \to \pm\infty} \cosh(x) = \infty$! We will see later that $\cos(z)$ attains every possible complex value infinitely often! But at the moment, the moral is just to be careful transferring properties of real trigonometric functions to their complex counterparts.

Question

What sort of beast is $\tan(z) = \frac{\sin(z)}{\cos(z)}$?

Solution.

The quotient rule says that it will be differentiable on its natural domain with $\frac{d}{dz}(\tan(z)) = \sec^2(z)$. But what is its natural domain? In the real case, we know that $\tan(x)$ is not defined at all $x = \frac{\pi}{2} + k\pi$ with $k \in \mathbf{Z}$. But $0 = \cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ if and only if $e^{iz} = -e^{-iz}$ or $e^{i2z} = -1$. But $-1 = e^{i\pi}$, so our earlier result tells us $\cos(z) = 0$ if and only if $i2z = i\pi + 2\pi ik$ for some $k \in \mathbf{Z}$. Thus $\cos(z) = 0$ if and only if $z = \frac{\pi}{2} + k\pi$ for some $k \in \mathbf{Z}$. Thus the complex zeros of $\cos(z)$ are the same as its real zeros. Thus $\tan(z)$ is analytic on $\mathbf{C} \setminus \{ \frac{\pi}{2} + k\pi : k \in \mathbf{Z} \}$. □

That's Enough for Now