The Complex Logarithm

Definition

If $z \in \mathbf{C} \setminus \{0\}$, then we define

$$\log(z) = \{\, w \in \mathbf{C} : e^{w} = z \,\}.$$

Remark

$$\log(z) = \{\, \ln(|z|) + i \arg(z) \,\} = \{\, \ln(|z|) + i\theta : \theta \in \arg(z) \,\}.$$

A Nice Formula

Theorem

Suppose that $zw \neq 0$. Then

$$\log(zw) = \log(z) + \log(w) = \{\, a + b : a \in \log(z) \text{ and } b \in \log(w). \,\}$$

Definition

We define the principal value of $\log(z)$ to be

$$\mathrm{Log}(z) = \ln(|z|) + i\,\mathrm{Arg}(z).$$

I will refer to the function $z \mapsto \mathrm{Log}(z)$ for all $z \neq 0$ as the principal branch of $\log(z)$. Notice that if $x > 0$, then $\mathrm{Log}(x) = \ln(x)$ so the principal branch extends the natural logarithm to D^*.

Analytic Branches of $\log(z)$

Theorem

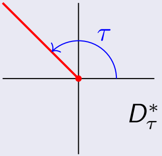
The function $g(z) = \mathrm{Log}(z)$ *is analytic in* $D^* = \mathbf{C} \setminus (-\infty, 0]$ *and* $g'(z) = \dfrac{d}{dz}\big(\mathrm{Log}(z)\big) = \dfrac{1}{z}$.

Remark

We have other analytic branches of $\log(z)$ such as

$$\mathcal{L}_\tau(z) = \ln\big(|z|\big) + i \arg_\tau(z).$$

which is analytic in $D^*_\tau = \mathbf{C} \setminus \{\, re^{i\tau} : 0 \le r < \infty \,\}$. Moreover $\frac{d}{dz}\big(\mathcal{L}_\tau(z)\big) = \frac{1}{z}$.



As Promised

Corollary

The function $u(z) = \ln\big(|z|\big)$ *is harmonic in the punctured plane* $\mathbf{C} \setminus \{0\}$. *Furthermore, the functions* $v_\tau(z) = \arg_\tau(z)$ *are harmonic in* D^*_τ. *Note that* $\operatorname{Arg}(z) = \arg_{-\pi}(z)$ *and hence is harmonic in* D^*.

Branches

Definition

Suppose that f is a set-valued function on a domain D. Then a continuous function F on D is called a branch of f on D if $F(z) \in f(z)$ for all $z \in D$.

Remark

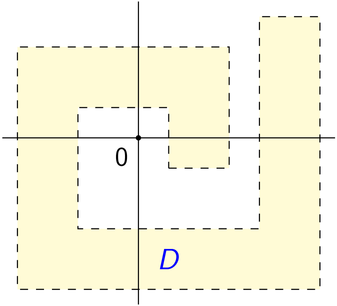
Thus a branch of f is just want we usually want: a continuous way of choosing one element of $f(z)$ for each $z \in D$.

Example

1. $F(z) = \text{Arg}(z)$ is a branch of $f(z) = \arg(z)$ in D^*.
2. $F(z) = \text{Log}(z)$ is an analytic branch of $f(z) = \log(z)$ in D^*. Similarly, $\mathcal{L}_\tau(z)$ is an analytic branch of $\log(z)$ in D^*_τ.
3. Note that $\text{Log}(z)$ and $\mathcal{L}_\pi(z)$ are both branches of $\log(z)$ in D^*.

An Instructive Example

There is an analytic branch of $\log(z)$ defined in all of the domain D drawn at right. Note that no single one of our crude branches $\mathcal{L}_\tau(z)$ suffices. We had to more clever. The real point here is not the formula we worked out, but that even complicated domains can admit analytic branches of $\log(z)$ even if there is no obvious way to write them down.



No Accident

Proposition

Suppose that g *is an analytic branch of* $\log(z)$ *in a domain* $D \subset \mathbf{C} \setminus \{0\}$*. Then* $g'(z) = \frac{1}{z}$*.*

Proposition

If f *and* g *are both analytic branches of* $\log(z)$ *in a domain* D*, then there is a* $k \in \mathbf{Z}$ *such that* $f(z) = g(z) + 2\pi i k$ *for all* $z \in D$*.*