Math 43: Spring 2020
Lecture 8 Part 1

Dana P. Williams

Dartmouth College

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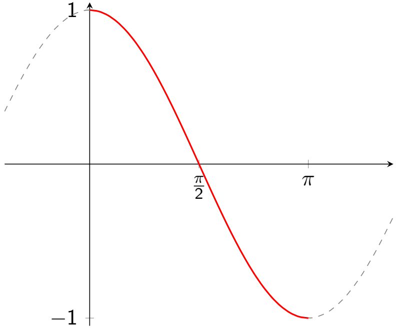
Back in the Day

Remark (Inverse Trig Functions)

Since sine and cosine are not one-to-one, we define inverse trigonometric functions by restricting the domain so that they become one-to-one:

$$y = \cos^{-1}(x) \quad \text{if and only if} \quad \cos(y) = x \quad \text{and} \quad y \in [0, \pi].$$



Today's Goal

Remark

The game today is to introduce a complex logarithm. This is easy in the real case since $x \mapsto e^x$ is one-to-one. However since $z \mapsto e^z$ is no longer one-to-one, we have the same sort of issues to deal with that we did back in the day with inverse trig functions!

The Complex Logarithm

Definition

If $z \in \mathbf{C} \setminus \{0\}$, then we define

$$\log(z) = \{\, w \in \mathbf{C} : e^w = z \,\}.$$

Remark

Just as with $\arg(z)$, $\log(z)$ is a set-valued function. But what set is it? Let $w = x + iy$ and try to solve $e^{x+iy} = z$. Let $z = re^{i\theta}$. Then we want

$$e^x e^{iy} = re^{i\theta}.$$

This means $e^x = r$ and hence $x = \ln(r) = \ln(|z|)$. But we also need $e^{iy} = e^{i\theta}$. Thus $y = \theta + 2\pi ik$ with $k \in \mathbf{Z}$.
Thus

$$\log(z) = \{\, \ln(|z|) + i\arg(z) \,\} = \{\, \ln(|z|) + i\theta : \theta \in \arg(z) \,\}.$$

An Example

Example

Find $\log(\sqrt{3}+i)$.

Solution.

The easy part is $|\sqrt{3}+i| = 2$. To compute the argument, note that $\cos(\theta) = \frac{\sqrt{3}}{2}$ and $\sin(\theta) = \frac{1}{2}$. Thus $\theta = \frac{\pi}{6} + 2\pi k$. Thus

$$\log(\sqrt{3}+i) = \{\, \ln(2) + i\tfrac{\pi}{6} + 2\pi i k : k \in \mathbf{Z} \,\}. \qquad \square$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$\cos(\theta)$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

A Nice Formula

Theorem

Suppose that $zw \neq 0$. *Then*

$$\log(zw) = \log(z) + \log(w) = \{\, a + b : a \in \log(z) \textit{ and } b \in \log(w).\, \}$$

Proof.

We have $\log(zw) = \{\, \ln\big(|zw|\big) + i \arg(zw) \,\} =$
$\{\, \log(|z|) + \log(|w|) + i(\arg(z) + \arg(w) \,\}$. And this is

$$\{\, \ln(|z|) + i \arg(z) \,\} + \{\, \ln(|w|) + i \arg(w) \,\} = \log(z) + \log(w).$$

$\square$

A Certain Preference For Actual Functions

Definition

We define the principal value of $\log(z)$ to be

$$\operatorname{Log}(z) = \ln(|z|) + i \operatorname{Arg}(z).$$

I will refer to the function $z \mapsto \operatorname{Log}(z)$ for all $z \neq 0$ as the principal branch of $\log(z)$. Notice that if $x > 0$, then $\operatorname{Log}(x) = \ln(x)$ so the principal branch extends the natural logarithm to D^*.

Example

$$\operatorname{Log}(-1 - i) = \frac{1}{2}\ln(2) - i\frac{3\pi}{4}.$$

Let's Take a Break