Math 43: Spring 2020
Lecture 8 Part 3

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Branches

Remark

We have seen a number of set-valued functions such as $z \mapsto \arg(z)$, $z \mapsto \log(z)$, and even $z \mapsto z^{\frac{1}{n}}$ for any $n \geq 2$. To my annoyance, the text calls these "multiple-valued functions". But our new techniques deal with complex-valued functions.

Definition

Suppose that f is a set-valued function on a domain D. Then a continuous function F on D is called a branch of f on D if $F(z) \in f(z)$ for all $z \in D$.

Remark

Thus a branch of f is just want we usually want: a continuous way of choosing one element of $f(z)$ for each $z \in D$.

Examples

Example

1. $F(z) = \mathrm{Arg}(z)$ is a branch of $f(z) = \arg(z)$ in D^*.
2. $F(z) = \mathrm{Log}(z)$ is an analytic branch of $f(z) = \log(z)$ in D^*. Similarly, $\mathcal{L}_\tau(z)$ is an analytic branch of $\log(z)$ in D^*_τ.
3. Note that $\mathrm{Log}(z)$ and $\mathcal{L}_\pi(z)$ are both branches of $\log(z)$ in D^*.

Example

$F(z) = \exp\bigl(\frac{1}{2}\mathrm{Log}(z)\bigr)$ is an analytic branch of $z^{\frac{1}{2}}$ in D^*: to see this, just notice that $F(z)^2 = \exp(2 \cdot \frac{1}{2}\mathrm{Log}(z)) = \exp(\mathrm{Log}(z)) = z$. Hence $F(z) \in z^{\frac{1}{2}}$ as required. Also notice that if $x > 0$, then $F(x) = \sqrt{x}$.

Pasting

Remark

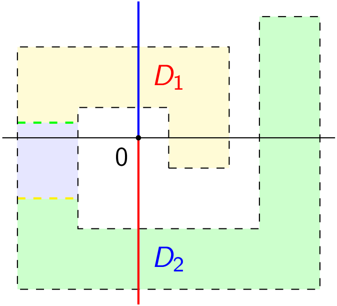
Since complex differentiablity is a local property we can build new analytic functions by pasting together analytic functions that agree on open overlaps. Suppose $f : D_1 \subset \mathbf{C} \to \mathbf{C}$ is analytic on a domain D_1 and $g : D_2 \subset \mathbf{C} \to \mathbf{C}$ is analytic on a domain D_2. We know from homework that $D = D_1 \cup D_2$ is a domain proved $D_1 \cap D_2$ is not empty. If $g(z) = f(z)$ for all $z \in D_1 \cap D_2$, then we can define an analytic function h on D by

$$h(z) = \begin{cases} f(z) & \text{if } z \in D_1, \text{ and} \\ g(z) & \text{if } z \in D_2. \end{cases}$$

This technique is called pasting. When you glue the pieces together, they have to overlap perfectly.

An Instructive Example

Let D be the domain which is the union of the domains D_1 and D_2 where D_1 consists of the yellow region and D_2 is the green region. The idea is the D_1 and D_2 overlap in the blue region. We want to find an analytic branch of $\log(z)$ in all of D. Note no $\mathcal{L}_\tau(z)$ will be analytic or even continuous in all of D. But $f(z) = \mathcal{L}_{-\frac{\pi}{2}}(z)$ is an analytic branch in D_1. Similarly, $g(z) = \mathcal{L}_{\frac{\pi}{2}}(z)$ is an analytic branch in D_2. Furthermore f and g are equal in the blue overlap!



Then $h(z) = \begin{cases} f(z) & \text{if } z \in D_1 \\ g(z) & \text{if } z \in D_2 \end{cases}$ is an analytic branch of $\log(z)$ in all of D!

No Accident

Proposition

Suppose that g is an analytic branch of $\log(z)$ in a domain $D \subset \mathbf{C} \setminus \{0\}$. Then $g'(z) = \dfrac{1}{z}$.

Proof.

Since by definition, $g(z) \in \log(z)$, we have $z = e^{g(z)}$. Since g is analytic by assumption, we can differentiate both sides to get $1 = g'(z)e^{g(z)} = g'(z) \cdot z$. Now divide both sides by z. $\square$

Branches of $\log(z)$

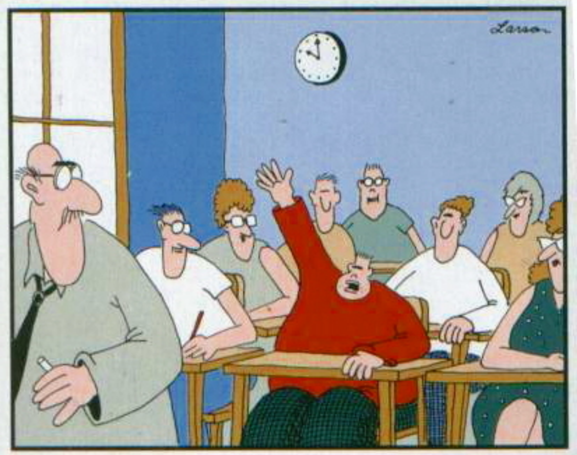
Question

Suppose that f and g are both analytic branches of $\log(z)$ in a domain D. How are f and g related?

Solution.

First, observe that by definition, $e^{f(z)} = z = e^{g(z)}$ for all $z \in D$. This is just what it means to be a branch of $\log(z)$! Thus if $z_0 \in D$, we have $e^{f(z_0)} = z_0 = e^{g(z_0)}$. Then $f(z_0) = g(z_0) + 2\pi i k_0$ for some $k_0 \in \mathbf{Z}$. Similarly, if $z_1 \in D$, then $f(z_1) = g(z_1) + 2\pi i k_1$ with $k_1 \in \mathbf{Z}$. We'd like to argue that $k_0 = k_1$ and that there is a fixed $k \in \mathbf{Z}$, that does not depend on z, such that $f(z) = g(z) + 2\pi i k$ for all $z \in D$. A hint as to how to prove this is the realization that we are trying to prove that $h(z) = f(z) - g(z)$ is a constant. But by the previous result, $h'(z) = f'(z) - g'(z) = \frac{1}{z} - \frac{1}{z} = 0$. Hence h is constant. Therefore any two analytic branches of $\log(z)$ in a domain D must differ by a constant multiple of $2\pi i$. $\square$

That is Enough for One Lecture



"Mr. Osborne, may I be excused? My brain is full."

With all due apologies to Gary Larson and to you.