Math 43: Spring 2020
Lecture 9 Summary

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Complex Powers

Lemma ($\alpha = \frac{1}{n}$)

If $z \neq 0$, then $z^{\frac{1}{n}} = \exp\big(\frac{1}{n}\log(z)\big)$.

Remark

Here $z^{\frac{1}{n}}$ is the set of n^{th}-roots of z. On the other hand, $\exp\big(\frac{1}{n}\log(z)\big)$ is the set of values $e^{\frac{1}{n}w}$ where $w \in \log(z)$. The lemma asserts these two sets are the same.

Definition

If $\alpha \in \mathbf{C}$ and $z \neq 0$, then we define $z^{\alpha} = \exp\big(\alpha\log(z)\big)$.

Remark

This is a reasonable definition only because z^n with $n \in Z$ and $z^{\frac{1}{n}}$ with $n \in \mathbf{N}$ still mean what they always did. Notice that unless α is an integer, then z^{α} is a set.

Moving On

Definition

A continuous function $z(t) = x(t) + iy(t)$ on a closed interval $[a, b] \subset \mathbf{R}$ is called a path. We say z is differentiable if x and y are. Then $z'(t) = x'(t) + iy'(t)$ is a tangent vector to the curve parameterized by z if $z'(t) \neq 0$.

Smooth Arcs and Smooth Curves

Definition

A smooth arc $\gamma \subset \mathbf{C}$ is the image of a differentiable path $z : [a, b] \to \mathbf{C}$ such that

(a) $z'(t) \neq 0$ for all $t \in [a, b]$, and

(b) z is one-to-one on $[a, b]$.

We call $\gamma \subset \mathbf{C}$ a smooth closed curve if γ is the image of a differentiable path $z : [a, b] \to \mathbf{C}$ such that

(c) $z'(t) \neq 0$ for all $t \in [a, b]$, and

(d) z is one-to-one on $[a, b)$ with $z(a) = z(b)$ and $z'(a) = z'(b)$.

A smooth curve is either a smooth arc or a smooth closed curve.

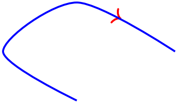
If γ is a smooth curve and $z : [a, b] \to \mathbf{C}$ is a functions satisfying either (a) and (b), or (c) and (d) above, then $z : [a, b] \to \mathbf{C}$ is called an admissible parameterization of γ.

Orientation

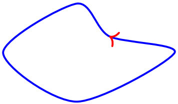
Remark

An admissible parameterization $z : [a, b] \to \mathbf{C}$ of a smooth curve γ gives γ an orientation. In the case of smooth arc, we move from $z(a)$ to $z(b)$. Thus if $a \leq t_1 < t_2 \leq b$, $z(t_1)$ comes before $z(t_2)$. In the case of a smooth closed curve, if $a < t_1 < t_2 < b$, again $z(t_1)$ comes before $z(t_2)$. (Things get fuzzy at the initial point $z(a)$ so don't worry about that.)

Orientations



(a) A Smooth Arc



(b) A Smooth Closed Curve

Remark

Note that a given smooth curve has two possible orientations. An admissible parameterization is compatible with one and only one of these.

More Terminology

Definition

A smooth curve together with an orientation is called a directed smooth curve. An admissible parameterization of a directed smooth curve is an admissible parameterization consistent with the orientation.

Remark

If $z : [a, b] \to \mathbf{C}$ is an admissible parameterization of a directed smooth curve γ, then so are

1. $w(t) = z\big(a + t(b - a)\big)$ for $t \in [0, 1]$, and
2. $\sigma(t) = z\big(a + \frac{b-a}{d-c}(t - c)\big)$ for $t \in [c, d]$.

This means that given a directed smooth curve, we can always find an admissible parameterization with any domain interval we wish. The usual choice is $[0, 1]$, but we are free to do as we please.

The Opposite Curve

Definition

If γ is a directed smooth curve, then we write $-\gamma$ for the set γ with the opposite orientation.

Example

Let $z : [a, b] \to \mathbf{C}$ be an admissible parameterization of a directed smooth curve γ. Then the following are admissible parameterizations of the opposite curve $-\gamma$.

1. $w : [-b, -a] \to \mathbf{C}$ given by $w(t) = z(-t)$.
2. $\sigma : [0, 1] \to \mathbf{C}$ given by $\sigma(t) = z\big(b + t(a - b)\big)$.