Math 43: Spring 2020
Lecture 9 Part 1

Dana P. Williams

Dartmouth College

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Remark

Now that we have a working relationship with the complex logarithm, we can play some games. For motivation, let's think back to real exponentials such as x^α for a real constant α. But how do we make sense of something like $2^{\sqrt{2}}$? We actually define $2^\alpha = e^{\alpha \ln 2}$. But we can do something similar over the complexes as we now have both a complex exponential and a complex logarithm.

n^{th}-Roots

Lemma ($\alpha = \frac{1}{n}$)

If $z \neq 0$, then $z^{\frac{1}{n}} = \exp\big(\frac{1}{n}\log(z)\big)$.

Remark

Here $z^{\frac{1}{n}}$ is the set of n^{th}-roots of z. On the other hand, $\exp\big(\frac{1}{n}\log(z)\big)$ is the set of values $e^{\frac{1}{n}w}$ where $w \in \log(z)$. The lemma asserts these two sets are the same.

Proof.

If $z = re^{i\theta}$, then

$$\begin{aligned}\exp\Big(\frac{1}{n}\log(z)\Big) &= \exp\Big(\frac{1}{n}(\ln(r) + i\theta + 2\pi ik)\Big)\\ &= \exp\Big(\frac{1}{n}\ln(r)\Big)\exp\Big(i\frac{\theta + 2\pi k}{n}\Big)\\ &= \sqrt[n]{r}\exp\Big(i\frac{\theta + 2\pi k}{n}\Big)\end{aligned}$$

and we get distinct values only for $k = 0, 1, 2, \dots, n-1$ which is exactly how we enumerated $z^{\frac{1}{n}}$ earlier. $\square$

Complex Powers

Definition

If $\alpha \in \mathbf{C}$ and $z \neq 0$, then we define $z^\alpha = \exp\big(\alpha \log(z)\big)$.

Remark

This is a reasonable definition only because z^n with $n \in Z$ and $z^{\frac{1}{n}}$ with $n \in \mathbf{N}$ still mean what they always did. Notice that unless α is an integer, then z^α is a set.

Example

Find i^i.

Solution.

$i^i = \exp\big(i \log(i)\big) = \exp\big(i(0 + i\frac{\pi}{2} + 2\pi i k)\big) = \exp(-\frac{\pi}{2} - 2\pi k) = e^{-\frac{\pi}{2}} e^{2\pi k}$ for $k \in \mathbf{Z}$. $\square$

Principal Branch

Definition

We call $F(z) = \exp\big(\alpha \operatorname{Log}(z)\big)$ the principal branch of z^{α} in D^*.

Example (Calculus)

Note that $F(z) = \exp(\alpha \operatorname{Log}(z))$ is analytic in D^*. Furthermore,
$F'(z) = \frac{\alpha}{z} \exp(\alpha \operatorname{Log}(z)) = \frac{\alpha}{z} F(z) =$
$\alpha \exp(-\operatorname{Log}(z)) \exp(\alpha \operatorname{Log}(z)) = \alpha \exp\big((\alpha - 1) \operatorname{Log}(z)\big)$.

Thus we are tempted to write

$$\frac{d}{dz}(z^{\alpha}) = \alpha z^{\alpha-1}.$$

But this only makes sense for functions. The above says it is valid if we argee to use the principal value on both sides of the displayed equation!

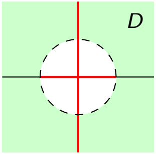
Branches are Tricky

Example

Consider $f(z) = (z^2 - 1)^{\frac{1}{2}}$. Can we find an analytic branch of f in $D = \{ z : |z| > 1 \}$?

We want to find an analytic function F in D such that $F(z)^2 = z^2 - 1$! An obvious first guess would be $F(z) = \exp\bigl(\frac{1}{2}\operatorname{Log}(z^2 - 1)\bigr)$. But where is F analytic? The natural domain of analyticity of F is $D' = \{ z : z^2 - 1 \in D^* \}$. Since $(x + iy)^2 - 1 = x^2 - y^2 - 1 + i(2xy)$, we see that $x + iy \notin D'$ when $xy = 0$ and $x^2 - y^2 - 1 \le 0$.

Thus if $x = 0$, then $x + iy \notin D'$. On the other hand, if $y = 0$, then $x + iy \notin D'$ if $|x| \le 1$. Thus D' is the complement of the imaginary axis and the segment $[-1, 1]$. Unfortunately, $D \not\subset D'$.



Let's Not Give Up

We could also try $G(z) = \exp\bigl(\mathcal{L}_0(z^2 - 1)\bigr)$. Then G is analytic in $D'' = \{\, z : z^2 - 1 \in \mathbf{C} \setminus [0,\infty) \,\}$. I leave it to you to check that D'' is the complement of the two rays $(-\infty, 1]$ and $[1, \infty)$. DO THIS! Sadly, we still have $D \not\subset D''$.

Fortunately, the authors of our text have a trick to share. Observe that $z^2 - 1 = z^2\bigl(1 - \frac{1}{z^2}\bigr)$. So we could try $H(z) = z\exp\bigl(\frac{1}{2}\operatorname{Log}(1 - \frac{1}{z})\bigr)$. Then H is analytic in $D''' = \{\, z : 1 - \frac{1}{z^2} \in D^* \,\}$. But $1 - \frac{1}{z^2} = 1 - \frac{1}{x^2 - y^2 + i2xy}$ is real only when $xy = 0$. If $x = 0$, then $1 - \frac{1}{-y^2} \geq 0$. If $y = 0$, then $1 - \frac{1}{x^2} \leq 0$ only when $|x| \leq 1$. Thus $D''' = \mathbf{C} \setminus [-1, 1]$ and $D \subset D'''$. Yay.

Example

Let $\cos^{-1}(z) = \{\, w \in \mathbf{C} : \cos(w) = z \,\}$. Describe this set.

If $\cos(w) = z$, then $z = \frac{1}{2}(e^{iw} + e^{-iw})$. Hence $e^{iw} - 2z + e^{-iw} = 0$, and $(e^{iw})^2 - 2ze^{iw} + 1 = 0$. Thus $e^{iw} = \frac{2z+(4z^2-4)^{\frac{1}{2}}}{2} = z + (z^2-1)^{\frac{1}{2}}$. (Keep in mind $(z^2-1)^{\frac{1}{2}}$ is 2-valued.) We can then take logarithms to get $iw = \log(z + (z^2-1)^{\frac{1}{2}})$ or $w = -i\log(z + (z^2-1)^{\frac{1}{2}})$. Thus we can write

$$\cos^{-1}(z) = -i\log(z + (z^2-1)^{\frac{1}{2}}).$$

Time for a Break