Math 43: Spring 2020
Lecture 9 Part 2

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Next Chapter

Now this is not the end.
It is not even the beginning of the end.
But it is, perhaps, the end of the beginning.

This quote is due to Winston Churchill from his address to the House of Commons after the Battle of El Alamein on November 10, 1942.

Of considerably less historical import, we have now covered all the material you will be responsible for on the preliminary exam. Recall that the exam will be made available Wednesday, April 22nd and must be uploaded to your canvas account by Friday, April 24th prior to the start of class on Friday. (That is, 10:10am EDT.)

Paths

We will have need of an important concept from multivarible calculus. Recall that a path in $\mathbf{C} = \mathbf{R}^2$ is a continuous function $z : [a, b] \to \mathbf{C}$. This might be more familiar if we write $z(t) = x(t) + iy(t)$ for continuous functions $x, y : [a, b] \to \mathbf{R}$. Then the image of z is what we call a parameterized curve. (Recall that $x(t) + iy(t)$ is just our notation for $\big(x(t), y(t)\big)$.) We say that z is differentiable if both $x'(t)$ and $y'(t)$ exist for all $t \in (a, b)$ and the one-sided derivatives exist at a and b. We let

$$z'(t) = x'(t) + iy'(t).$$

Tangent Vectors

If $z'(t_0) \neq 0$, then we call $z'(t_0)$ the tangent vector to the curve at the point $z(t_0)$.

If we think of $z(t)$ as the position of a particle at time t, then $z'(t_0)$ indicates the instantaneous direction of the particle moving with speed $|z'(t_0)|$.

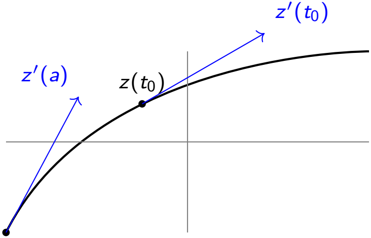
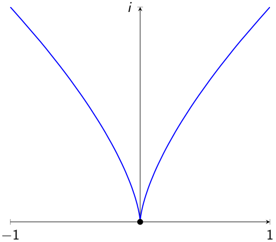


Figure: A Parametrized Curve

Smooth Curves

Now let $z(t) = t^3 + it^2$ for $t \in [-1,1]$. Note that $z'(0) = 0$. The curve parametrized by z is the same as the graph of $y = x^{\frac{2}{3}}$ over $[-1,1]$. We don't want to call this curve smooth since it has a cusp at $(0,0)$!

Clearly, there is no well-defined tangent to the curve at $(0,0)$. But consider the curve $w(t) = t^3 + it^6$ for $t \in [-1,1]$. Again, $w'(0) = 0$. However this curve is the same as the graph of $y = x^2$ over $[-1,1]$. But this curve should be called smooth as it has a well defined tangent at each point.



Smooth Arcs and Smooth Curves

Definition

A smooth arc $\gamma \subset \mathbf{C}$ is the image of a differentialbe path $z : [a, b] \to \mathbf{C}$ such that

(a) $z'(t) \neq 0$ for all $t \in [a, b]$, and

(b) z is one-to-one on $[a, b]$.

We call $\gamma \subset \mathbf{C}$ a smooth closed curve if γ is the image of a differentiable path $z : [a, b] \to \mathbf{C}$ such that

(c) $z'(t) \neq 0$ for all $t \in [a, b]$, and

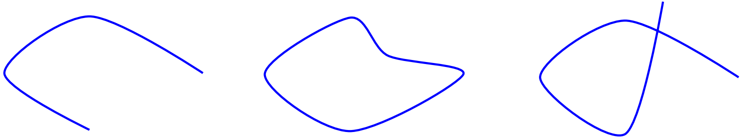
(d) z is one-to-one on $[a, b)$ with $z(a) = z(b)$ and $z'(a) = z'(b)$.

A smooth curve is either a smooth arc or a smooth closed curve.

If γ is a smooth curve and $z : [a, b] \to \mathbf{C}$ is a functions satisfying either (a) and (b), or (c) and (d) above, then $z : [a, b] \to \mathbf{C}$ is called an admissible parameterization of γ.

Examples

Some Examples



(a) A Smooth Arc (b) A Smooth Closed Curve (c) Not a Smooth Curve

Example

Let $\gamma = [w_1, w_2]$ with $w_1 \neq w_2$. Then $z(t) = w_1 + t(w_2 - w_1)$ with $t \in [0, 1]$ is an admissible parameterization of γ.

Example

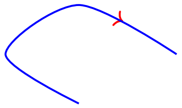
Let $\gamma = \{\, z : |z - z_0| = r \,\}$. Then $z(t) = z_0 + re^{it}$ with $t \in [0, 2\pi]$ is an admissible parameterization of γ.

Orientation

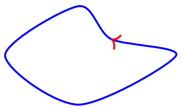
Remark

An admissible parameterization $z : [a, b] \to \mathbf{C}$ of a smooth curve γ gives γ an orientation. In the case of smooth arc, we move from $z(a)$ to $z(b)$. Thus if $a \leq t_1 < t_2 \leq b$, $z(t_1)$ comes before $z(t_2)$. In the case of a smooth closed curve, if $a < t_1 < t_2 < b$, again $z(t_1)$ comes before $z(t_2)$. (Things get fuzzy at the initial point $z(a)$ so don't worry about that.)

Orientations



(a) A Smooth Arc



(b) A Smooth Closed Curve

Remark

Note that a given smooth curve has two possible orientations. An admissible parameterization is compatible with one and only one of these.

More Terminology

Definition

A smooth curve together with an orientation is called a directed smooth curve. An admissible parameterization of a directed smooth curve is an admissible parameterization consistent with the orientation.

Remark

If $z : [a, b] \to \mathbf{C}$ is an admissible parameterization of a directed smooth curve γ, then so are

1. $w(t) = z\big(a + t(b - a)\big)$ for $t \in [0, 1]$, and
2. $\sigma(t) = z\big(a + \frac{b-a}{d-c}(t - c)\big)$ for $t \in [c, d]$.

This means that given a directed smooth curve, we can always find an admissible parameterization with any domain interval we wish. The usual choice is $[0, 1]$, but we are free to do as we please.

The Opposite Curve

Definition

If γ is a directed smooth curve, then we write $-\gamma$ for the set γ with the opposite orientation.

Example

Let $z : [a, b] \to \mathbf{C}$ be an admissible parameterization of a directed smooth curve γ. Then the following are admissible parameterizations of the opposite curve $-\gamma$.

1. $w : [-b, -a] \to \mathbf{C}$ given by $w(t) = z(-t)$.
2. $\sigma : [0, 1] \to \mathbf{C}$ given by $\sigma(t) = z\big(b + t(a - b)\big)$.

That's Enough for This Week.