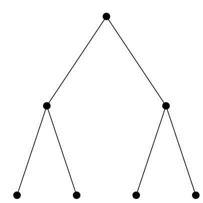
Math 68. Algebraic Combinatorics.

Problem Set 4. Due on Tuesday, 11/27/2007.

1. How many necklaces (up to cyclic symmetry) have n read beads and n blue beads? (Express your answer as a sum over all divisors d of n.)

2. Let Γ be the graph shown below.



   An automorphism of Γ is a permutation π of the vertices of Γ that preserves adjacencies (i.e., there is an edge between two vertices x and y if and only if there is an edge between $\pi(x)$ and $\pi(y)$). Let G be the automorphism group of Γ, so G has order 8.

   (a) What is the cycle index polynomial of G, acting on the vertices of Γ?

   (b) In how many ways can one color the vertices of Γ in n colors, up to symmetry of Γ?

3. For any finite group G of permutations of an ℓ-element set X, let $f(n)$ be the number of inequivalent (under the action of G) colorings of X with n colors. Find $\lim_{n\to\infty} f(n)/n^\ell$. Interpret your answer as saying that "most" colorings of X are asymmetric (have no symmetries).

4. Consider the group G of (orientation-preserving) symmetries of the cube.

   (a) Show that $|G| = 24$.

   (b) Find the number of inequivalent colorings of the faces of the cube using n colors.

   (c) Find the number of inequivalent colorings of the vertices of the cube using n colors.

5. Let $c(\lambda)$ denote the number of corner squares (or distinct parts) of the partition λ. For instance, $c(5,5,4,2,2,2,1,1) = 4$. Show that

$$\sum_{\lambda \vdash n} c(\lambda) = p(0) + p(1) + \cdots + p(n-1),$$

   where $p(i)$ denotes the number of partitions of i (with $p(0) = 1$).