Algebra Exam 1
Due Wednesday, February 20

1   Let $F \subseteq E$ be a Galois field extension. Assume that $G = \mathrm{Gal}(E/F)$ is a simple group (recall that a simple group is one that has no proper normal subgroups besides the trivial subgroup).

(a) Let $\alpha \in E$. If $\alpha \notin F$, show that E is a splitting field for $m_{F,\alpha}$.

(b) Let $H \subseteq G$ be a proper subgroup. Show that E is a splitting field over F for some polynomial $f \in F[X]$ with $\deg(f) = |G : H|$.

(c) Assume further that G is nonabelian. Show that all the roots of unity in E (i.e. all $\alpha \in E$ such that $\alpha^n = 1$ for some n) actually lie in F.

2   Let $F \subseteq E$ be a field extension. Suppose K and L are intermediate fields with compositum $J = \langle K, L \rangle$. Assume that K and L are both Galois over F.

(a) Show that J is Galois over F.

(b) If $\mathrm{Gal}(K/F)$ and $\mathrm{Gal}(L/F)$ are both abelian, show that $\mathrm{Gal}(J/F)$ is also abelian,

(**Hint:** For (b), recall that the set of all commutators in G generates a subgroup $G' \subseteq G$ with the property that G/H is an abelian group if and only if $G' \subseteq H$. Now determine the fixed field of the commutator subgroup of $\mathrm{Gal}(J/F)$.)

3   Let $F \subseteq E$ be an algebraic extension of fields. For this problem only, define

- $\alpha \in E$ is **Galois** over F if $F[\alpha] \supseteq F$ is a Galois field extension;

- $\alpha \in E$ is **Abelian** over F if α is Galois over F and $\mathrm{Gal}(F[\alpha]/F)$ is an abelian group.

Answer the questions True or False. If your answer is True, prove it. If your answer is False, give a counterexample.

(a) The set $K = \{\alpha \in E \mid \alpha \text{ is Galois}\}$ is a field.

(b) The set $K = \{\alpha \in E \mid \alpha \text{ is Abelian}\}$ is a field.

(**Hint:** For (a), do you know of any specific elements α in some field that are *not* Galois? Can you think of a Galois extension without a primitive element?)