CHAPTER 1
Matrix Algebra

In this chapter, we shall study the concept of a matrix, one of the fundamental tools of mathematics. You may have seen matrices in connection with the solution of equations, and we explain exactly how that application of matrices works. However, matrices have fascinating applications in the theory of graphs, in geometry, in multivariable calculus, and even in probability. Matrices also provide an excellent data structure for use in computer programs. We shall see how the answers to certain practical questions can be found by applying standard arithmetic operations on matrices.

We begin our study of matrices with a study of their arithmetic and some of the algebraic properties this arithmetic has. Next, we make a careful study of the role of matrices in solving systems of equations. We develop a technique that can be used by a person or programmed onto a computer to determine all solutions to any system of linear equations. Our study of equation-solving leads us to the concept of an invertible matrix and the inverse of such a matrix. This idea has intellectual appeal, since it extends ideas from arithmetic and ordinary algebra, as well as vast practical significance. Many practical applications of invertibility require more knowledge than we can provide here. In order to determine whether matrices are invertible, we introduce the concept of a determinant. The reader may have been introduced to the time-consuming and detailed application of determinants to the solution of systems of two or three linear equations in two or three unknowns. Since we have more efficient methods for solving equations, we do not pursue this application, but rather the application of determining invertibility.

Section 1-1
Matrix Arthmetic

A Sums and Numerical Products

A horizontal list of numbers such as

$$[1,2,3] \qquad [-x,y] \qquad [2+x,3-y,2x+z,y-z] \qquad \text{or} \qquad [3]$$

is often called a **row matrix**, and a vertical list of numbers such as

$$\begin{bmatrix} x-y \\ z+2x \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \qquad \begin{bmatrix} 10 \\ 20 \\ 0 \\ 30 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -17 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 0.5 \\ 0.66 \\ 17 \\ 3.53 \end{bmatrix}$$

is often called a **column matrix**. A rectangle of numbers such as

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} \qquad \begin{bmatrix} 10 & 20 & 30 \\ 5 & 25 & 30 \end{bmatrix} \qquad \begin{bmatrix} 0.1 & 2.3 \\ 3.4 & 0.5 \\ 1.1 & 5.5 \\ 6 & 1 \end{bmatrix}$$

is called a **matrix**. The first matrix immediately above is called a two-by-two matrix; the other three are a three-by-three, a two-by-three, and a four-by-two matrix, respectively. An **m-by-n (or $m \times n$)matrix** consists of m rows of n numbers each. We normally will enclose the entries of a matrix in rectangular braces as above. Row matrices and column matrices are often called *row vectors* and *column vectors* respectively. We use a slightly different notation than the notation we use for vectors because the kinds of things we do with matrices are sometimes different from what we do with vectors.

We can use matrices (the plural of *matrix*) to keep track of related quantities. For example, if a store is ordering small, medium, large, and jumbo eggs, it might use the column matrix

$$\begin{bmatrix} 10 \\ 20 \\ 80 \\ 30 \end{bmatrix} = Q$$

to keep track of the fact that it is ordering 10 dozen small eggs, 20 dozen medium eggs, 80 dozen large eggs, and 30 dozen jumbo eggs. Often we use an upper-case letter to denote a matrix: here we have called our matrix Q to stand for quantity.

Matrix Addition

We form the **sum** of two m-by-n matrices by adding corresponding entries. We form the **difference** by subtracting corresponding entries. For example,

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3 & -2 \end{bmatrix}$$

EXAMPLE 1 Suppose a farmer has the following orders for dozens of eggs from three local stores.

	Store 1	Store 2	Store 3
Sm	10	20	20
Med	20	30	40
Lg	80	160	100
Jum	30	40	40

How many of each type of egg should the farmer load on the truck to deliver the egg orders?

Solution The matrix sum

$$\begin{bmatrix} 10 \\ 20 \\ 80 \\ 30 \end{bmatrix} + \begin{bmatrix} 20 \\ 30 \\ 160 \\ 40 \end{bmatrix} + \begin{bmatrix} 20 \\ 40 \\ 100 \\ 40 \end{bmatrix} = \begin{bmatrix} 50 \\ 90 \\ 340 \\ 110 \end{bmatrix}$$

tells us how many of each type of eggs should be on the truck. ■

Note that we have only defined the sum for two matrices that are *both* m-by-n matrices. The rule we used to define the sum is the same kind of rule we used to define the sum of vectors; to add two vectors we add corresponding entries as well. In fact, vectors are just special kinds of matrices.

EXAMPLE 2 If A, B, C, and D are the matrices below, which of the sums $A+B$, $B+C$, and $C+D$ are defined?

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Solution We say that two matrices have the same shape if they have the same number of rows and the same number of columns. A and B do not have the same shape, so $A+B$ is *not* defined. B and C do not have the same shape, so $B+C$ is *not* defined. C and D *do* have the same shape, so $C+D$ *is* defined. ■

There are rules for addition, such as $a+b=b+a$ (the commutative law) and $(a+b)+c = a+(b+c)$ (the associative law), that we use without consciously thinking about them when we do arithmetic. Fortunately, because matrix addition is just repeated numerical addition, these standard laws of addition apply to matrix addition as well.

Numerical Multiplication

To multiply a matrix M by a real number r, we multiply each entry of M by r.

EXAMPLE 3 If the wholesale prices for small, medium, large, and jumbo eggs (by the dozen) are given by the matrix $\begin{bmatrix} .60 & .70 & .80 & .90 \end{bmatrix}$, what matrix would represent the prices after a 10% increase?

Solution Increasing each price by 10% is the same as multiplying each price by 1.1, so our new price matrix would be

$$(1.1)P = 1.1 \begin{bmatrix} .60 & .70 & .80 & .90 \end{bmatrix} = \begin{bmatrix} .66 & .77 & .88 & .99 \end{bmatrix}$$

■

Notice that adding $-1 \cdot M$ to M gives the all-zeros matrix which we denote by 0 and call the *zero matrix*, so $-1 \cdot M$ is the matrix we would naturally think of as $-M$.

Row-Column Products

There is an important operation called *multiplication of matrices*, which may be applied in a wide variety of problems. We begin our study of this operation with an example. We have seen that a store ordering small, medium, large, and jumbo eggs might use the column matrix

$$Q = \begin{bmatrix} 10 \\ 20 \\ 80 \\ 30 \end{bmatrix}$$

to keep track of the fact that they are ordering 10 dozen small eggs, 20 dozen medium eggs, 80 dozen large eggs, and 30 dozen jumbo eggs. Let us assume the egg farmer uses the matrix

$$P = \begin{bmatrix} .60 & .70 & .80 & .90 \end{bmatrix}$$

of Example 2 for the prices of small, medium, large, and jumbo eggs.

With these prices, how much would the order for eggs cost the store? We have to add up the prices times the quantities for each size of egg to get

$$\begin{aligned} \text{Cost} &= .60 \cdot 10 + .70 \cdot 20 + .80 \cdot 80 + .90 \cdot 30 \\ &= 6.00 + 14.00 + 64.00 + 27.00 \\ &= 111.00 \end{aligned}$$

In effect, we are multiplying the price matrix times the quantity matrix, giving

$$\text{Cost } = P \cdot Q$$

There is a way to define multiplication of matrices so that with this multiplication rule the cost matrix will be the price matrix times the quantity matrix. In particular, if R is a row matrix with n entries $r_1, r_2, \ldots, r_n$ and C is a column matrix with n entries $c_1, c_2, \ldots, c_n$, then we define the **product** RC by

$$[r_1, r_2, \ldots, r_n] \cdot \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} = R \cdot C = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

Thus when we multiply a row matrix on the left by a column matrix on the right, we get a number. We can, if we wish, think of this number as a matrix with just one entry, namely $[r_1c_1 + r_2c_2 + \cdots + r_nc_n]$.

If it looks to you like we are taking the dot product of the row matrix (vector) of prices with the column matrix (vector) of prices, you are right on the mark. The reason that we restrict our work with matrices to multiply a row matrix times a column matrix is that in a short time this "bookkeeping" system will make it possible for us to define a meaningful and useful concept of multiplication for more general matrices.

In the example that follows, we use $R1$ and $R2$ rather than R_1 and R_2 to stand for two row matrices, because there is no possibility of confusion between $R1$ and its first entry r_1.

EXAMPLE 4 Using the matrices

$$R1 = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \qquad C1 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \qquad C2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$
$$R2 = \begin{bmatrix} 5 & 4 & 2 & 3 \end{bmatrix}$$

compute $R1C1$ and $R2C2$. Explain why the product $R1C2$ cannot be computed.

Solution $R1C1 = 1\cdot 3 + 2\cdot 2 + 4\cdot(-1) = 3$; $R2C2 = 5\cdot 1 - 4\cdot 1 - 2\cdot 1 + 3\cdot 1 = 2$. The product $R1C2$ cannot be computed because $R1$ has 3 entries and $C2$ has 4 entries, and we cannot compute a product of a row matrix by a column matrix unless they have the same number of entries. ■

If it looks to you like we are taking the dot product of the row matrix (vector) of prices with the column matrix (vector) of prices, you are right on the mark. The reason that we organize our work to multiply a row matrix times a column matrix is that using this this system will make it possible for us to define a meaningful and useful concept of multiplication for more general matrices. Thus we will not talk about dot products in this section in order not to create confusion later on.

B Matrix Products

The *row-column product* described is a special case of a more general multiplication on matrices. To show how this more general operation arises, we expand our egg farmer example. The prices of eggs may change each week. In Table 1, we give (hypothetical) wholesale prices of eggs over a four-week period. The matrix that follows the table is obtained from the table by deleting the labels from the horizontal rows and vertical columns of the table and placing rectangular brackets around the body of the table.

Table 1

	Sm	Med	Lg	Jum
Week 1	.60	.70	.80	.90
Week 2	.55	.65	.75	.85
Week 3	.60	.75	.85	.95
Week 4	.65	.70	.85	.95

$$\begin{bmatrix} .60 & .70 & .80 & .90 \\ .55 & .65 & .75 & .85 \\ .60 & .75 & .85 & .95 \\ .65 & .70 & .85 & .95 \end{bmatrix} = P$$

As the table shows, each row of the matrix represents a different week of prices for eggs.

Let us now assume that the farmer deals with three stores and that each store has a standard weekly order for eggs, as shown in Table 2. The matrix following the table is simply the body of the table. We now use Q to stand for this matrix.

Table 2

	Store 1	Store 2	Store 3
Sm	10	20	20
Med	20	30	40
Lg	80	160	100
Jum	30	40	40

$$\begin{bmatrix} 10 & 20 & 20 \\ 20 & 30 & 40 \\ 80 & 160 & 100 \\ 30 & 40 & 40 \end{bmatrix} = Q$$

Now row i (which we denote by Ri) of Table 1 (matrix P) tells us the prices of eggs in week i. On the other hand, column j (which we denote by Cj) of Table 2 tells us the (standing) weekly order of eggs for store j. Thus if we form the product of the row matrix Ri with the column matrix Cj, the result is the cost in week i of the order for eggs from store j. Thus, for example, $R2C3$ is the cost in week 2 of the order from store 3. In order to keep track of our farmer's weekly billings, we should form the matrix in which the entry in row i and column j is the bill in week i for store j; we have seen that this is the product of Ri with Cj. For example, the entry in row 2 and column 3 will be the product of $R2$ and $C3$. This leads to the following definition.

The General Matrix Product

The **product** of a matrix M with rows $R1, R2, \ldots, Rm$, each of length n, times a matrix N with columns $C1, C2, \ldots, Ck$, each of length n, is the matrix whose entry in row i and column j is $Ri \cdot Cj$.

Symbolically, we write

$$M \cdot N = \begin{bmatrix} \text{---}R1\text{---} \\ \text{---}R2\text{---} \\ \vdots \\ \text{---}Ri\text{---} \\ \vdots \\ \text{---}Rm\text{---} \end{bmatrix} \cdot \begin{bmatrix} | & | & & | & & | \\ C1 & C2 & \cdots & Cj & \cdots & Ck \\ | & | & & | & & | \end{bmatrix}$$

$$= \begin{array}{cc} \begin{array}{c} \text{column } j \\ \downarrow \end{array} & \\ \begin{bmatrix} R1\cdot C1 & \cdots & R1\cdot Cj & \cdots & R1\cdot Ck \\ \vdots & & \vdots & & \\ Ri\cdot C1 & \cdots & Ri\cdot Cj & \cdots & Ri\cdot Ck \\ \vdots & & \vdots & & \\ Rm\cdot C1 & \cdots & Rm\cdot Cj & \cdots & Rm\cdot Ck \end{bmatrix} & \leftarrow \text{row } i \end{array}$$

In particular,

$$\begin{bmatrix} \text{---}R1\text{---} \\ \text{---}R2\text{---} \end{bmatrix} \begin{bmatrix} | & | & | \\ C1 & C2 & C3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} R1C1 & R1C2 & R1C3 \\ \\ R2C1 & R2C2 & R2C3 \end{bmatrix}$$

(This shows why we first defined the product only for a row matrix on the left by a column matrix on the right. That definition helps us remember that the entry in *row* i and *column* j of the product MN is *row* i of M times *column* j of N.)

EXAMPLE 5 If $M = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$, compute the product MN and explain why the product NM cannot be computed.

Solution We highlight a typical entry, the entry in row 2 and column 3, as well as the row and column it came from, in bold.

$$\begin{aligned} MN &= \begin{bmatrix} 1 & 2 \\ \mathbf{2} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} 3 & -2 & \mathbf{1} \\ 1 & 2 & \mathbf{-3} \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot 3 + 2\cdot 1 & -2\cdot 1 + 2\cdot 2 & 1\cdot 1 - 2\cdot 3 \\ 2\cdot 3 - 1\cdot 1 & -2\cdot 2 - 1\cdot 2 & \mathbf{2}\cdot\mathbf{1} + (\mathbf{-1})(\mathbf{-3}) \end{bmatrix} \\ &= \begin{bmatrix} 3+2 & -2+4 & 1-6 \\ 6-1 & -4-2 & \mathbf{2}+\mathbf{3} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -5 \\ 5 & -6 & \mathbf{5} \end{bmatrix} \end{aligned}$$

The product NM cannot be computed because the rows of N have three entries and the columns of M have two entries. ■

EXAMPLE 6 If P and Q are the matrices below, compute $P \cdot Q$ and $Q \cdot P$.

$$P = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

Solution

$$\begin{aligned} P \cdot Q &= \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1+4 & -1+4 \\ -2+6 & 2+6 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 4 & 8 \end{bmatrix} \\ Q \cdot P &= \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1+2 & 2-3 \\ 2-4 & 4+6 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 10 \end{bmatrix} \end{aligned}$$

■

Example 6 shows one of the more interesting features of matrix multiplication. The commutative law $P \cdot Q = Q \cdot P$ that holds for multiplication of numbers *need not work* for multiplication of matrices. (In fact it *usually does not* work for matrix multiplication.)

EXAMPLE 7 Using the matrix I shown below and the matrices P and Q of Example 6, compute IP and QI.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution We write

$$IP = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0(-2) & 1 \cdot 2 + 0 \cdot 3 \\ 0 \cdot 1 + 1(-2) & 0 \cdot 2 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$$

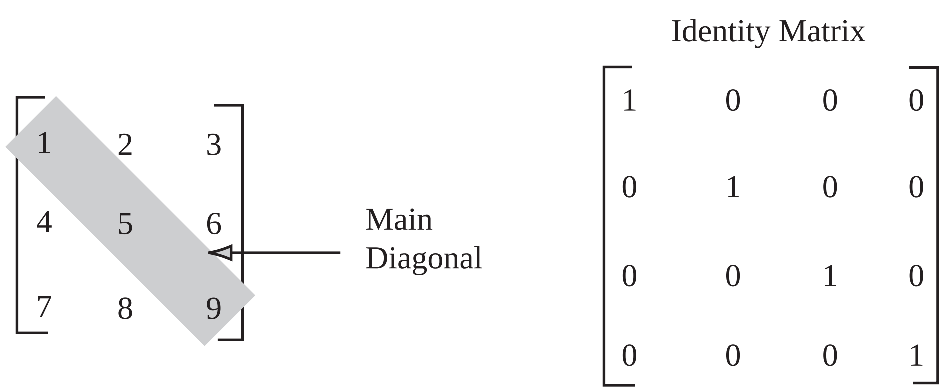
We also write

$$QI = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + (-1) \cdot 0 & 1 \cdot 0 + (-1) \cdot 1 \\ 2 \cdot 1 + 2 \cdot 0 & 2 \cdot 0 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

Thus $IP = P$ and $QI = Q$. ■

In Example 7, we see an analogy between matrix multiplication and numerical multiplication that *does* work. The matrix I behaves like the number one. We can describe this analogy more generally with a bit more terminology. An n-by-n matrix is called a **square matrix**. The **main diagonal** of an n-by-n matrix consists of the entries in row 1 and column 1, row 2 and column 2, and in general row i and column i for each i from 1 to n. (See Figure 1.) An **identity matrix** is a square matrix with ones as its main diagonal entries and zeroes as all its other entries. Thus the matrix I in Example 7 is a two by two identity matrix. I is a standard symbol for an identity matrix of any size. It follows from the defintion of matrix multiplication that $MI = M$ and $IM = M$ whenever the products are defined.

Figure 1

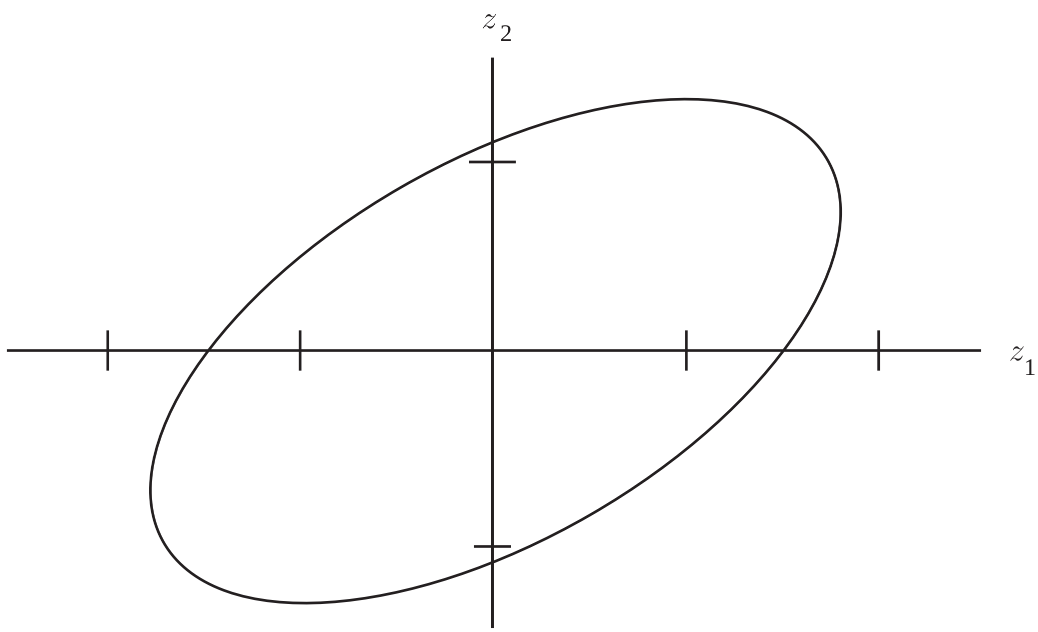


Fortunately, as we shall see later, two other important laws of arithmetic, the associative law and the distributive law do hold for matrix multiplication.

Using Matrices in Geometry

So far we have talked about only one coordinate system for two and three dimensional space. However in trigonometry courses and precalculus courses you learn about how to rotate the axes of a two dimensional space to convert equations into simpler form. Instead of talking about x and y axes, we will talk about x_1 and x_2 axes, y_1 and y_2 axes, and z_1 and z_2 axes. To get used to this new way of labelling our axes, suppose that we start with a coordinate system in which

Figure 2



the z_1 axis is the horizontal axis,, and the z_2 axis is the vertical axis. Suppose we want to identify the graph of the equation

$$13z_1^2 - 6\sqrt{3}z_1z_2 + 7z_2^2. \tag{1}$$

When we draw it carefully, we get an graph like that in Figure 2. The graph appears to be an ellipse, but its major axis is at an angle to the z_1 axis; perhaps a 30 degree angle.

This suggests that we might put in a new set of axes at a 30 degree angle to the old one and see how they fit the picture. We do so in Figure 3. This figure strongly suggests that we really do have an ellipse and that its equation in the x_1 and x_2 coordinate system is $x^2/4 + y^2 = 1$. To check this we have to see how to convert this equation into an equation in the z_1–z_2 coordinate system.

In Figure 4 we illustrate why it is that if we have an x_1 and x_2 axis that make an angle of θ with the z_1 and z_2 axes respectively, then we have the two equations

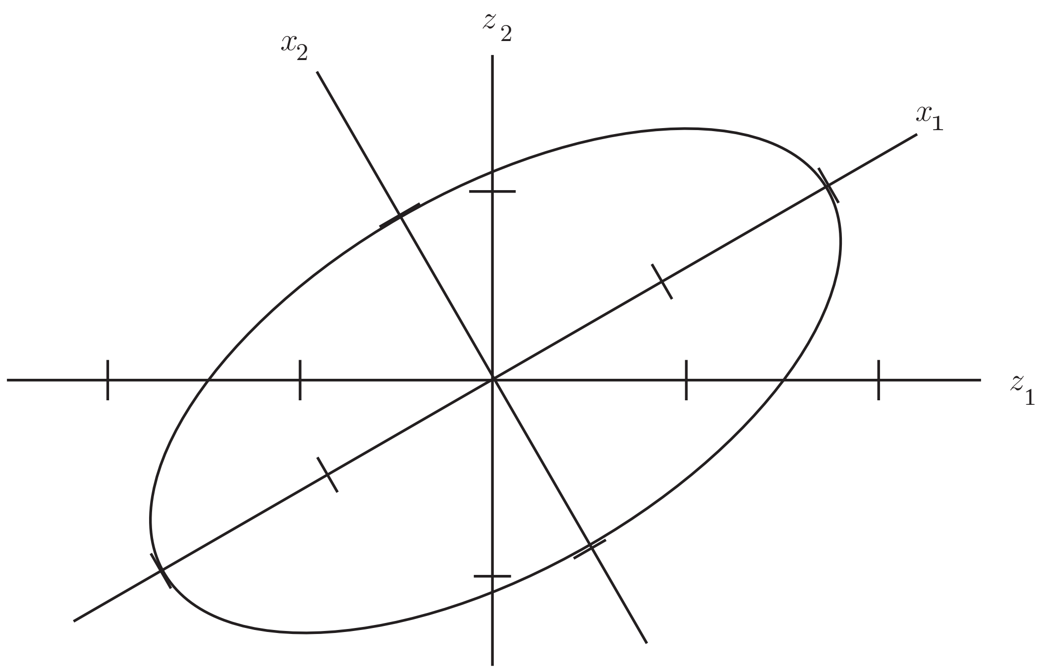
$$\begin{aligned} z_1 &= x_1 \cos\theta - x_2 \sin\theta \\ z_2 &= x_1 \sin\theta + x_2 \cos\theta. \end{aligned}$$

Notice that these two equations combine into the single matrix equation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

EXAMPLE 8 What is the matrix by which we must multiply $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in order to get the z_1 and z_2 coordinates if we rotate the x-axes through 30^o clockwise to get the z–axes?

Figure 3



Solution We must multiply by the matrix

$$\begin{bmatrix} \cos\pi/6 & -\sin\pi/6 \\ \sin\pi/6 & \cos\ pi/6 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

■

EXAMPLE 9 What is the equation in x_1–x_2 coordinates of the curve described in Equation 1 in z_1–z_2 coordinates?

Solution From Example 8 we see that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2 \\ \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \end{bmatrix}.$$

Thus on substitution for z_1 and z_2 the left hand side of Equation 1 becomes

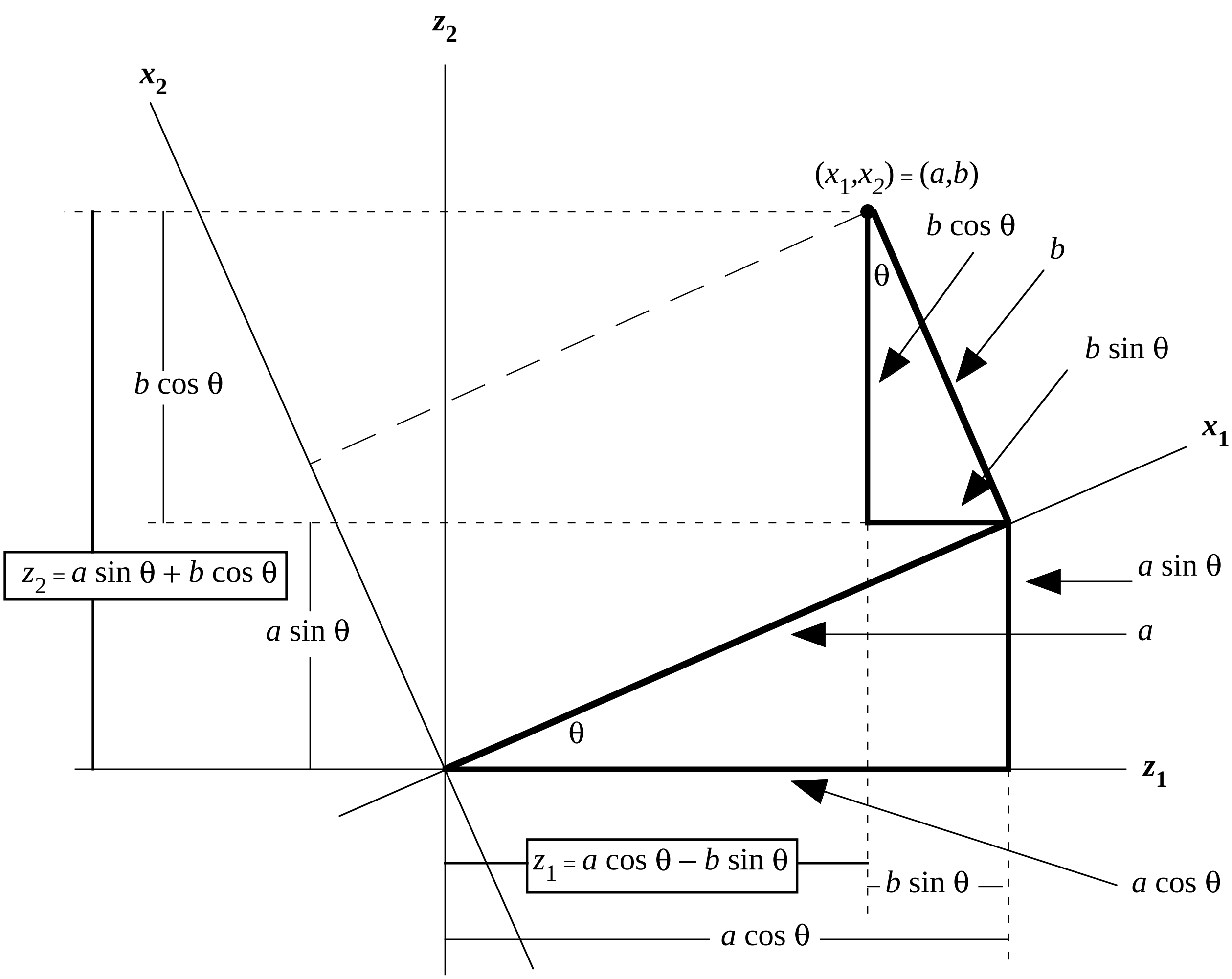
$$7(\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2)^2 - 6\sqrt{3}(\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2)(\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2) + 13(\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2)^2$$
$$= 4x_1^2 + 16x_2^2$$

Thus in the x_1–x_2 coordinate system, our ellipse has the equation $4x_1^2 + 16x_2^2 = 16$, which in standard form is

$$\frac{x_1^2}{4} + x_2^2 = 1.$$

This is the equation of an ellipse with major axis 2 and minor axis 1, exactly as we predicted. ■

Figure 4 The formulas for rotating coordinate systems



EXAMPLE 10 Suppose we first rotate our coordinate system through an angle θ clockwise to get from the x_1 and x_2 axes to the z_1 and z_2 axes, and then rotate another angle φ clockwise to get from the z_1 and z_2 axes to the y_1 and y_2 axes. What matrices must we multiply times $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in order to get the value for the matrix $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ of y-coordinates? What is this product of two matrices and what does it tell us about the sine and cosine of a sum of two angles?

Solution Since $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and since $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, we must (by the associative law) multiply $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by the product

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi\cos\theta - \sin\varphi\sin\theta & -\cos\varphi\sin\theta - \sin\varphi\cos\theta \\ \sin\varphi\cos\theta + \cos\varphi\sin\theta & -\sin\varphi\sin\theta + \cos\varphi\cos\theta \end{bmatrix}.$$

This shows that

$$\cos(\varphi + \theta) = \cos\varphi\cos\theta - \sin\varphi\sin\theta$$

and

$$\sin(\varphi + \theta) = \sin\varphi\cos\theta + cos\varphi\cos\theta.$$

■

A function defined by $f(X) = MX$ from the set of column vectors of length n to the set of column vectors of length m (so that M is an m by n matrix) is called a **linear transformation**. Since matrix multiplication is associative (see problem 2 at the end of this section for a hint of how to prove this), if $g(Y) = NY$ is a linear transformation from the set of column vectors of length m to the set of column vectors of length k, then the composition $g \circ f$ of the functions g and f is a linear transformation and is given by $g \circ f(X) = (NM)X$. The previous example illustrates exactly this situation. The composition of a rotation (of the coordinate system) through an angle θ and a rotation through an angle φ is a rotation through an angle $\varphi + \theta$, and the matrix that represents this composition is the product of the two matrices representing the individual rotations.

Double-Subscript Notation

It is inconvenient to speak of "the element in row i and column j of the matrix M" to refer to this entry. In **double-subscript notation** for a matrix M, people use either

$$M_{ij} \qquad \text{or} \qquad m_{ij}$$

(read as "em sub eye jay" or "em eye jay") to stand for the entry in row i and column j of a matrix M. There are times when an uppercase letter M is most appropriate and times when a lowercase letter m is most appropriate; usually it is a matter of choice. Thus a matrix M with three rows and four columns would be written

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \end{bmatrix}$$

EXAMPLE 11 In the matrix M below, what are M_{21} and M_{13}?

$$M = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 6 & 4 \\ 2 & -3 & 1 \end{bmatrix}$$

Solution M_{21} is -2, the element in row 2 and column 1, and M_{13} is 5, the element in row 1 and column 3. ■

If M has rows $R1, R2, \ldots$ and N has columns $C1, C2, \ldots$, then we would use double subscript notation to write the definition of the product MN as

$$(MN)_{ij} = RiCj$$

which says, "the i, j entry of MN is the product of row i of M with column j of N." We can also write

$$(MN)_{ij} = \sum_{k=1}^{n} M_{ik}N_{kj},$$

which is a more explicit way to describe the product of row i with column j. It is this form of the definition of matrix multiplication that you will usually see in more advanced courses both in mathematics and in other subjects.

EXAMPLE 12 Write down the double-subscript representation for a 2-by-2 matrix M, a 2-by-2 matrix N, and the product MN.

Solution We write

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} = \begin{bmatrix} m_{11}n_{11} + m_{12}n_{21} & m_{11}n_{12} + m_{12}n_{22} \\ m_{21}n_{11} + m_{22}n_{21} & m_{21}n_{12} + m_{22}n_{22} \end{bmatrix}$$

■

Note that in Example 12 the element $(MN)_{12}$ is given by $m_{11}n_{12} + m_{12}n_{22}$, which is what we get by expanding the notation

$$\sum_{k=1}^{n} m_{1k}n_{k2}$$

We sometimes need (or find it more convenient) to use the double-subscript notation and the double-subscript formula for matrix multiplication when we make symbolic computations, as for example in the proof of the **distributive law for matrix multiplication**.

Theorem 1 If A and B are m-by-n matrices and C is an n-by-p matrix, then $(A + B)C = AC + BC$.

Proof We compute the entry in row i and column j of $(A+B)C$ and of $(AB+AC)$. First we use the fact that $(A+B)_{ik} = A_{ik} + B_{ik}$ to write

$$((A+B)C)_{ij} = \sum_{k=1}^{n}(A+B)_{ik}C_{kj} = \sum_{k=1}^{n}(A_{ik}+B_{ik})C_{kj} = \sum_{k=1}^{n} A_{ik}C_{kj} + \sum_{k=1}^{n} B_{ik}C_{kj}$$

Second, we use the fact that $(AC+BC)_{ij} = (AC)_{ij} + (BC)_{ij}$ and the double subscript formula for matrix multiplication to write

$$(AC+BC)_{ij} = (AC)_{ij} + (BC)_{ij} = \sum_{k=1}^{n} A_{ik}C_{kj} + \sum_{k=1}^{n} B_{ik}C_{kj}$$

This shows that $((A+B)C)_{ij} = (AC+BC)_{ij}$ for each i and j, so the matrices $(A+B)C$ and $AC+BC$ have exactly the same entries. Therefore they are equal. ■

EXAMPLE 13 Using the matrices M, P, and Q of Examples 5 and 6, compute the product $(M+P)Q$ and the sum $MQ+PQ$ to show that they are equal.

Solution

$$\begin{aligned}
(M+P)Q &= \left(\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\right)\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 4 & 4 \end{bmatrix} \\
MQ+PQ &= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 4 & 4 \end{bmatrix}
\end{aligned}$$

■

The other major law of arithmetic, the **associative law of multiplication**, $(MN)P = M(NP)$, may also be proved by using the double-subscript notation and the double-subscript formula for matrix multiplication. Problem 2 gives a hint of how this works out.

Concepts Review

1. An array of numbers in a rectangle is called a(n) ____________ .
2. To find the ____________ of two matrices we add the corresponding entries.
3. An m-by-n matrix has ____________ rows and ____________ columns.
4. The product of a row matrix with entries r_1 through r_n times a column matrix with entries c_1 through c_n is given by the formula $R \cdot C =$ ____________ .
5. To multiply an m-by-n matrix times an r-by-s matrix we must have ____________ equal to ____________ .

6. The entry in row i and column j of the matrix product MN is the product of __________ __________ of M with __________ __________ of N.
7. In double-subscript notation, m_{ij} stands for the entry in __________ i and __________ j of the matrix M.
8. The __________ __________ of an n-by-n matrix A consists of A_{11}, A_{22}, ..., A_{nn}.
9. A(n) __________ matrix has ones on the main diagonal and zeros off the main diagonal and when multiplied by the matrix A gives the result __________ .
10. The formula $\sum_{j=1}^{n} A_{ij}B_{jk}$ is a formula which allows us to compute the entry in row __________ and column __________ of the matrix __________ .

A. Exercises

For Exercises 1 through 46 below, use the matrices

$$R = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} \quad E = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

1. Find $R + S$ and $R - T$.
2. Find $T + S$ and $R - S$.
3. Verify that $(R + S) + T = R + (S + T)$.
4. Verify that $(C + D) + E = C + (D + E)$.
5. Find the sum $S + O$ and explain what this illustrates.
6. Find the sum $O + R$ and explain what this illustrates.

In Exercises 7 through 46 below, use the matrices above and also the matrices

$$L = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

In Exercises 7 through 14, state whether the given sum is defined.

7. $L + M$
8. $M + N$
9. $L + N$
10. $L + Q$
11. $P + Q$
12. $R + T$
13. $M + C + D$
14. $R + T + S + P$
15. Find $L - N$ and $L + N$.
16. Find $N + Q$ and $Q - N$.
17. Find the product RC.
18. Find the product RD.
19. Find the product SD.
20. Find the product TC.

B. Exercises

For Exercises 21 through 46 below, use the matrices in Part A above.

22. Find the product PC.
23. Find the product PD.
24. N^2 means N times N. Find N^2 and N^4.
25. Find L^2 and L^4.
26. Find the product $P(C+E)$ and the sum $PC+PE$. What does this illustrate?
27. Find the product $P(D+E)$ and the sum $PD+PE$. What does this illustrate?

In Exercises 27 through 32, which of the products are defined?

27. MC	28. PC	29. QE
30. MD	31. QC	32. PE

33. Find the product MN. Is the product NM defined?
34. Find the product PM. Is the product MP defined?
35. Find the product NQ and the product QN. What does this illustrate?
36. Find the product LQ and the product QL. What does this illustrate?
37. Find the products $(LN)Q$ and $L(NQ)$. What does this illustrate?
38. Find the products $(TM)L$ and $T(ML)$. What does this illustrate?
39. Find the product $(R+S)M$ and the sum $RM+SM$. What does this illustrate?
40. Find the product $L(N+Q)$ and the sum $LN+LQ$. What does this illustrate?
41. Write down the 3-by-3 identity matrix I and compute the product IP.
42. Write down the 4-by-4 identity matrix I and compute the product PI.
43. What is P_{13}? What is P_{31}?
44. What is M_{12}? What is M_{21}?
45. Find the value of the following.

$$\sum_{k=1}^{2} M_{1k}Q_{k2}$$

    In what row and column of MQ do we find this entry?

46. Find the value of the following.

$$\sum_{k=1}^{4} P_{3k}m_{k2}$$

    In what row and column of PM do we find this entry?

Problems

1. Prove the distributive law $A(B+C) = AB + AC$ for two-by-two matrices.

2. Prove the associative law $A(BC) = (AB)C$ for all matrices A, B, and C such that the sums and products are defined. (*Hint*: The i, j entry of either side may be shown to be
$$\sum_{k=1}^{n}\sum_{h=1}^{m} a_{ik}b_{kh}c_{hj}$$
or a similar (and equal) sum.)

3. Suppose A is a two-by-two matrix such that $AB = BA$ for *all* two-by-two matrices B. What can you say about A?

4. The cancellation rule—If $AB = AC$ and $A \neq 0$ then $B = C$—works for numbers. Give an example to show that it does not work for matrices.

5. Find an example of a matrix A with $A \neq \pm I$ but $A^2 = I$.

6. Show that the cancellation rule—If $A + C = A + D$ then $C = D$—holds for matrix addition.

7. Suppose M and N are matrices such that MN is defined. Show that row i of MN is row i of M multiplied times the matrix N.

8. Suppose that M and N are matrices such that MN is defined. Show that column j of MN is the matrix M multiplied by column j of N.

9. Prove that if N is an n-by-k matrix and I is an n-by-n identity matrix, then $IN = N$.

10. Prove by induction that $I^n = I$.

11. Using A^3 in the usual way, show that
$$(A^3)_{ij} = \sum_{k=1}^{n}\sum_{h=1}^{n} A_{ik}A_{kh}A_{hj}$$

12. For numbers, we have $(x+1)^2 = x^2 + 2x + 1$ and $(x+y)^2 = x^2 + 2xy + y^2$. Do these formulas hold for matrices (with 1 replaced by the n by n identity matrix and x and y replaced by n by n matrices)? Why or why not?

Section 1-2 Matrices and Systems of Linear Equations

A Matrix Representations of Systems of Equations

One reason for the importance of matrix multiplication is that it helps us write a system of linear equations as a matrix equation.

EXAMPLE 14 Rewrite the following systems of equations as matrix equations.

(a)
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 + x_3 &= 6 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 + x_3 &= 6 \\ 2x_1 - x_2 + x_3 &= 3 \end{aligned}$$

Solution (a) Each individual equation may be written using a product of a row matrix and a column matrix. For example, $x_1 - 2x_2 + x_3 = 0$ may be written as

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

and $2x_1 + x_2 - x_3 = 1$ and $x_1 + x_2 + x_3 = 6$ may be written as

$$\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \qquad \text{and} \qquad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 6$$

The definition of the product of a rectangular matrix times a column matrix given in Section 10–1 shows that these three equations may be written as the single matrix equation

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

(b) Notice that this system of equations has all three equations from (a) and also the equation $2x_1 - x_2 + x_3 = 3$, which may be written as

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 3$$

Thus the second system may be written as

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 6 \\ 3 \end{bmatrix}$$

Notice how the matrix on the left follows exactly the pattern of the coefficients from the system of equations. ■

Solving Equations

You are probably familiar with the *elimination* or the *elimination and substitution* method of solving a system of equations. In either method, we add multiples of one equation to or subtract multiples of one equation from another with the goal of getting an equation involving one variable. In the pure elimination method, we continue this process until each variable lies in an equation by itself. In the elimination and substitution method, we get just one variable in an equation by itself, solve for it, substitute that value into other equations, and then repeat the process to find the values of the other variables. The pure elimination method may be combined with the use of matrices to give a method which is not only fast and effective for solving the system of equations but is especially easy to implement in a computer program.[1]

We review several ideas from algebra before giving an example of pure elimination. A **solution** to a system of equations in variables $x_1, x_2, \ldots, x_n$ is an assignment of numbers to the variables which makes each equation a true statement about numbers. For example, $x_1 = 1$, $x_2 = 2$, $x_3 = -1$ and $x_1 = 1$, $x_2 = -1$, $x_3 = 2$ are both solutions to the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$$

EXAMPLE 15 Check that $x_1 = 1$, $x_2 = 2$, $x_3 = -1$ is a solution to the equations above.

Solution We substitute 1 for x_1, 2 for x_2, and -1 for x_3 in the left-hand side and simplify, giving

$$\begin{aligned} 1 + 2 + (-1) &= 2 \\ 1 - 2 - (-2) &= 0 \end{aligned}$$

Therefore $x_1 = 1$, $x_2 = 2$, $x_3 = -1$ satisfies both equations and so is a solution. ■

We may check that $x_1 = 1$, $x_2 = -1$, $x_3 = 2$ is a solution similarly. In matrix terms, we would say that

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \text{and} \qquad X = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

are both solutions to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

EXAMPLE 16 Use matrix multiplication to verify that $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is a solution to the matrix equation $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

---

[1] While the elimination and substitution method is actually faster, it is slightly less easy to implement in a computer program (though the added speed might be worth the effort) and it does not have the nice theoretical implications that the pure elimination method has. Thus we will restrict ourselves to pure elimination here.

Solution We substitute the given column matrix for X, giving

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 & + & 1\cdot(-1) & + & 1\cdot 2 \\ 1\cdot 1 & + & (-1)(-1) & + & (-1)\cdot 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1-1+2 \\ 1+1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

■

We call the set of all column matrices X satisfying a matrix equation the **solution set** of the matrix equation or system of equations.

We say two systems of equations are **equivalent** if they have exactly the same solution sets. Notice this is the same thing we mean by saying the corresponding statements are equivalent. The two most standard ways of converting a system of equations to an equivalent system are

1. Multiply each term of one equation by the same nonzero real number r, or
2. Add a multiple of one equation to or subtract a multiple of one equation from a second one, replacing the second equation by that sum.

We shall see many examples of these operations in Example 17. The reason why each of these operations cannot change the solution set is that each is reversible. These two operations with equations will correspond to two operations we shall perform on rows of matrices in order to solve matrix equations. Let us illustrate the pure elimination method of solving systems of equations and show its effect on the matrix equations that correspond to them. Another operation that people sometimes use is to multiply one equation by a number and add a second equation to it. We are not going to consider this as one of our basic operations. Since this operation can be achieved by first multiplying an equation by a number and then subtracting another one from it, do it in those two steps when you really think that is the best approach.

EXAMPLE 17 Solve the first system of equations from Example 14, showing the complete system of equations and its matrix representation as you go.

Solution We write

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 1 \\ x_1 + x_2 + x_3 &= 6 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

Now we subtract twice the first equation from the second and write the result in place of the second equation giving

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 0 + 5x_2 - 3x_3 &= 1 \\ x_1 + x_2 + x_3 &= 6 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & -3 & 1 \\ 0 & 5 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

Now subtract the first equation from the third, giving

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 5x_2 - 3x_3 &= 1 \\ 3x_2 \qquad &= 6 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

Now we divide the last equation by 3, giving

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 5x_2 - 3x_3 &= 1 \\ x_2 \qquad &= 2 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Next we add twice the third equation to the first and subtract five times the third equation from the second giving

$$\begin{aligned} x_1 + x_3 &= 4 \\ -3x_3 &= -9 \\ x_2 &= 2 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ 2 \end{bmatrix}$$

Now we divide the second equation by -3, giving

$$\begin{aligned} x_1 + x_3 &= 4 \\ x_3 &= 3 \\ x_2 &= 2 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Next, we subtract the second equation from the first, giving

$$\begin{aligned} x_1 &= 1 \\ x_3 &= 3 \\ x_2 &= 2 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Finally, to list our solution in order, we interchange equations 2 and 3, giving

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In this way, we see that our system of equations has the solution $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. ■

B Row Reduced Matrices

The Coefficient Matrix

In Example 17, we showed a matrix version of each stage of the solution process. By concentrating on how the solution process influenced the matrices, we will be able to develop a method of solving equations which is suitable for use on a computer and is also convenient to do "by hand."

Notice that although the matrix on the left, which we shall call the **coefficient matrix**, changed as we worked through the equations and the matrix on the right, sometimes called the matrix of constants, changed as we worked through the equations, the matrix of variables didn't change a bit. Thus we might as well not have bothered to include the matrix of variables until the last step. Notice also that our operations first gave us exactly one *non*zero entry in the first column of the coefficient matrix, then exactly one *non*zero entry in the second column of the coefficient matrix and finally exactly one *non*zero entry in the third column of the coefficient matrix. Each operation we performed on the equations corresponds to what we call an *elementary row operation* on both the coefficient matrix and the right-hand side matrix. The three **elementary row operations** are

1. Multiply row i by a nonzero number: the **row multiple operation**.
2. Add a numerical multiple of one row to another: the **row sum operation**.
3. Interchange two rows: the **row interchange operation**

Each operation on the equations in which we multiplied an equation by a number corresponded to a row multiple operation. Each operation in which we added a multiple of one equation to another corresponded to a row sum operation on our coefficient and right-hand side matrices. Interchanging two equations corresponded to a row interchange operation.

Pivotal Entries

The fact that each row of the final matrix corresponded to the value of a different variable can be best explained in matrix terms by using the concepts of pivotal entries of a matrix and row reduced matrices.

The first nonzero entry of a row is called a **pivotal entry**.

A matrix is called **row reduced** if each *column* containing a pivotal entry has all its other entries equal to zero, and each pivotal entry is one.

Figure 5 shows a row reduced matrix and one that is not row reduced. In the first matrix there are two pivotal entries in the first column, and only one of them is one. In the second column of the matrix, there is only one pivotal entry, but it is not 1, and it is not the only nonzero entry in its column. In the third column, there is a pivotal entry, and all other entries in the column are zero. However the pivotal entry is not 1. Note that there are no pivotal entries in the last column. On the other hand, in the second matrix, each of the boldface ones is a pivotal entry. It is a one, and each other entry of its column is zero.

Figure 5

Not row reduced

$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & -4 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Row reduced

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 2 \\ 0 & \mathbf{1} & 2 & 0 & 0 & 3 \end{bmatrix}$$

The process in Example 17 of isolating all the variables into equations by themselves put the coefficient matrix into row reduced form. It is no surprise, then, that the idea of row reduced form turns out to be the foundation for our method of solving equations. What we shall do is reverse the emphasis of Example 17: We shall convert a system of equations to a matrix equation, perform row operations to put the coefficient matrix into row reduced form, and then write out the solutions to the equation on the basis of the row reduced form. This will save us from continually writing the variables, which is not only annoying but is a possible source of error.

Putting a Matrix into Row Reduced Form

For now, we shall concentrate on row operations on matrices. We shall discuss the relationship with equations later. We say a matrix M and a matrix N are **row equivalent** if there is a sequence of elementary row operations which converts M to N.

The following algorithm allows us to row reduce a matrix by imitating the pure elimination method of solving equations.

Algorithm Reduce
Input: An m-by-n matrix M.
Output: A row reduced m-by-n matrix which is row equivalent to M.
Procedure: In the left most column containing a pivotal entry choose one pivotal entry m_{ij}.
Divide Row i by m_{ij}.
For each $k \neq i$, subtract m_{kj} times row i from row k.
Repeat this process from left to right on all columns containing pivotal entries.

Theorem 2 Applying Algorithm Reduce to a matrix M yields a row reduced matrix which is row equivalent to M.

Proof When the process is complete, each column with a pivotal entry has only one nonzero entry, and this entry is one. Thus the matrix is row reduced; since all operations performed are row operations, the resulting matrix is row equivalent to M. ∎

The process described in algorithm Reduce is called **row reduction**. The pivotal entry we choose in a column is called a **pivot**. The algorithm gives us no instructions as to which pivotal entry we should choose (if there are several pivotal entries to choose form in a column). In a course in numerical analysis, you may study in detail how to choose the most appropriate pivot. For our purposes, no special choices are necessary.

EXAMPLE 18 Row reduce the matrix

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ -2 & -1 & 1 & 1 \end{bmatrix}$$

Solution We shall use an arrow underneath a symbol such as $\frac{1}{2} \cdot R1$ which stands for "multiply row 1 by $\frac{1}{2}$" or $(-1)R1 + R2$ which stands for "add negative one times row 1 to row 2," to describe which row operations are being performed as we row reduce our matrix. We put a symbol above and below an arrow when the arrow represents two operations. We will blindly follow the algorithm (described before Theorem 5); you may see some places where doing things in a slightly different order would save us from dealing with fractions or otherwise make our work easier. When you understand the process, it is appropriate for you to make such minor modifications in your own work, as we shall do in later examples.

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ -2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}\cdot R1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 2 & 1 & 2 & 3 \\ -2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow[2R1+R3]{(-2)R1+R2}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 4 & 5 \end{bmatrix} \xrightarrow{-1\cdot R2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 5 \end{bmatrix} \xrightarrow[-4R2+R3]{-\frac{3}{2}R2+R1}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[-1R3+R2]{-\frac{1}{2}R3+R1} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■

From the example, you see that some columns might not contain pivotal entries—the $\frac{1}{2}$ in the second column of the example is not pivotal. In our example, each row contained a pivotal entry. A row of zeros would be entirely possible in a row reduced matrix; a row of zeros cannot have a pivotal entry but, by definition, any row of nonzero entries must contain a pivotal entry. (Why?)

C Solving Systems of Equations

In part A of this section, we learned how to associate matrices with systems of equations and saw how operations on equations translated into operations on matrices. In part B, we saw how these operations can be used to row reduce a matrix. Let us state precisely what we have already learned about the relationship between matrices and systems of equations, so that we may use this knowledge as a basis for solving systems of equations.

The Relationship Between Matrices and Systems of Equations

Theorem 3 Let M be an m-by-n matrix and B be a column matrix with m entries. If we perform exactly the same row operations on M and B resulting in a row reduced m-by-n matrix N and column matrix C, then the system of equations represented by the matrix equation $MX = B$ and the system of equations represented by the matrix equation $NX = C$ are equivalent.

Proof This theorem simply summarizes what we already know. ■

In some cases row reduction makes the solutions to a system of equations obvious.

EXAMPLE 19 Solve the system of equations

$$\begin{aligned} 2x - y + z &= 1 \\ x + 2y - 2z &= 3 \\ 3x - 2y - z &= 4 \end{aligned}$$

Solution We shall rewrite the system as the matrix equation

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Now to make sure we perform the same operations on both the coefficient matrix and right-hand side matrix, we shall write them together in a single matrix with a dividing line between them:

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 1 & 2 & -2 & 3 \\ 3 & -2 & -1 & 4 \end{array}\right]$$

This is called the **augmented matrix** of the system of equations. (If the matrix to the left of the vertical line is M and to the right of the line is B, we use the symbol $M|B$ to stand for the augmented matrix.)

Now we perform row operations on the augmented matrix until we have the part to the left of the line row reduced. Notice that we do not blindly follow algorithm Reduce, though we follow the pattern it suggests.

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 1 \\ 1 & 2 & -2 & 3 \\ 3 & -2 & -1 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & -2 & 3 \\ 3 & -2 & -1 & 4 \end{array}\right] \xrightarrow[(-3)R1+R3]{(-1)R1+R2}$$

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{5}{2} & \frac{5k}{2} \end{array}\right] \xrightarrow[2R3]{\frac{2}{5}R2} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 1 \\ 0 & -1 & -5 & 5 \end{array}\right] \xrightarrow[1R2+R3]{\frac{1}{2}R2+R1}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -6 & 6 \end{array}\right] \xrightarrow{-\frac{1}{6}R3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{1R3+R2}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

What does this row reduced augmented matrix tell us? By Theorem 3, we know that our original system of equations has exactly the same solutions as the new system of equations that corresponds to the final augmented matrix. When we write this new system down, we get

$$\begin{aligned} x \qquad\qquad &= 1 \\ y \qquad &= 0 \\ z &= -1 \end{aligned}$$

Since this system of equations is equivalent to the original system, we see that the assignment $x = 1$, $y = 0$, $z = -1$, and only that assignment, makes all three equations true, and so we have solved our system of equations. ■

Row Reduction Always Determines All Solutions

The process we just demonstrated always provides solutions to our equations, though this fact is not always so obvious.

EXAMPLE 20 Solve the system of equations

$$\begin{aligned} 2x_1 + 1x_2 + 3x_3 + 4x_4 &= 0 \\ 2x_1 + 1x_2 + 2x_3 + 3x_4 &= 0 \\ -2x_1 - 1x_2 + 1x_3 + 1x_4 &= -1 \end{aligned}$$

Solution We write down the augmented matrix

$$\left[\begin{array}{rrrr|r} 2 & 1 & 3 & 4 & 0 \\ 2 & 1 & 2 & 3 & 0 \\ -2 & -1 & 1 & 1 & -1 \end{array}\right]$$

Since the left-hand part of the augmented matrix is the matrix of Example 18, we perform the row operations of that example on this augmented matrix to get

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

This corresponds to the system of equations

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 \qquad\qquad &= \tfrac{1}{2} \\ x_3 \qquad &= 1 \\ x_4 &= -1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= \tfrac{1}{2} - \tfrac{1}{2}x_2 \\ x_3 &= 1 \\ x_4 &= -1 \end{aligned}$$

Because each of our row operations is reversible, this system of equations has exactly the same solutions as the original system. Further if we choose some real number r and set $x_2 = r$, then our equations read $x_1 = \frac{1}{2} - \frac{1}{2}r$, $x_2 = r$, $x_3 = 1$, $x_4 = 1$. Thus whether we pick $r = \pi$, or $r = -\frac{3}{5}$, or $r = 10$, or $r = \sqrt{2}$, or anything else, these four equations, $x_1 = \frac{1}{2} - \frac{1}{2}r$, $x_2 = r$, $x_3 = 1$, $x_4 = 1$, give a solution to our system of equations. ■

Our system of equations in Example 20 has infinitely many solutions, and by letting r stand for an arbitrary real number (often called a **parameter**), we can express our solutions in terms of r. If we write down our solutions as a vector

$$(x_1, x_2, x_3, x_4) = (\frac{1}{2} - \frac{1}{2}r, r, 1, 1),$$

we can see that

$$(x_1, x_2, x_3, x_4) = (\frac{1}{2}, 0, 1, 1) + r(-\frac{1}{2}, 1, 0, 0).$$

This is the equation of the line in four-dimensional space that goes through the point $(\frac{1}{2}, 0, 1, 1)$ and is parallel to the vector $(-\frac{1}{2}, 1, 0, 0)$.

Example 20 suggests the fundamental theorem which follows. We say the variable x_i is **pivotal** if column i contains a pivotal entry; in Example 20, variables x_1, x_3, and x_4 were pivotal for the row reduced augmented matrix.

Theorem 4 If a system of equations has the matrix representation $MX = B$, then forming the augmented matrix $M|B$ and row reducing it to an augmented matrix $N|C$ with N in row reduced form allows us to determine the solutions to the system of equations as follows.

1. Write down the equations corresponding to the matrix equation $NX = C$.
2. If any of these equations has the form $0 = r$ where $r \neq 0$, then the original system of equations has no solutions.
3. Set each nonpivotal variable equal to a different **parameter** (which means a symbol standing for an arbitrary real number) and express the pivotal variables in terms of these parameters. The set of column matrices described by these expressions is the solution set to the system of equations.

Proof Since the row reduced system has the same solutions as the original system, there is nothing left to prove. ■

EXAMPLE 21 Find all solutions to the system of equations

$$\begin{aligned} x + y &= 1 \\ x + y &= 2 \end{aligned}$$

Solution The augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

and row reduces to

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

corresponding to $x + y = 1$ and $0 = 1$. Since $0 = 1$ is impossible, the system has *no* solutions. ■

Having the Same Number of Unknowns as Equations Tells Us Nothing

Common examples of three equations in three unknowns have just one solution, that is, one value each for x, y, and z. This is not always the case.

EXAMPLE 22 Solve the system of equations

$$\begin{aligned} x - y + 2z &= 2 \\ x + y - 3z &= -1 \\ 5x + y - 5z &= 1 \end{aligned}$$

Solution We write

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 1 & 1 & -3 & -1 \\ 5 & 1 & -5 & 1 \end{array}\right] \xrightarrow[-5R1+R3]{-1R1+R2}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 0 & 2 & -5 & -3 \\ 0 & 6 & -15 & -9 \end{array}\right] \xrightarrow{-3R2+R3} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 0 & 2 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R2} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 2 \\ 0 & 1 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{1R2+R1} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since this is row reduced and variables x and y are the pivotal variables, we take z to be an arbitrary real number r and write down the equations

$$\begin{array}{rcl} x - \frac{1}{2}r & = & \frac{1}{2} \\ y - \frac{5}{2}r & = & -\frac{3}{2} \end{array}$$

After solving for x and y, we get

$$x = \frac{1}{2} + \frac{1}{2}r, \qquad y = -\frac{3}{2} + \frac{5}{2}r, \qquad z = r$$

so our system has infinitely many solutions. Again our solutions form a line, in this case we may represent our vector of solutions as

$$(x, y, z) = (\frac{1}{2} + \frac{1}{2}r, -\frac{3}{2} + \frac{5}{2}r, r) = (\frac{1}{2}, -\frac{3}{2}, 0) + r(\frac{1}{2}, \frac{5}{2}, 1).$$

This is the parametric representation for the line in three-dimensional space through the point $(\frac{1}{2}, -\frac{3}{2}, 0)$ and parallel to the vector $(\frac{1}{2}, \frac{5}{2}, 1)$. ■

Of course, there is no reason why answers should only involve one parameter.

EXAMPLE 23 Find all solutions to the system of equations

$$\begin{array}{rrrrl} x_1 & +2x_2+ & 2x_3 & -x_4 & = 1 \\ x_1 & +2x_2+ & x_3 & & = 2 \\ 2x_1 & +4x_2+ & 3x_3 & -x_4 & = 3 \end{array}$$

Solution We apply row reduction, following the general pattern of the row reduction algorithm (but not applying it blindly) to get

$$\left[\begin{array}{rrrr|r} 1 & 2 & 2 & -1 & 1 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 4 & 3 & -1 & 3 \end{array}\right] \xrightarrow[-2R1+R3]{-1R1+R2} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[2R2+R1]{-1R2+R3} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-1R2} \left[\begin{array}{rrrr|r} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Now x_2 and x_4 are our non-pivotal variables. We let them equal r and s respectively, and we write

$$\begin{aligned} x_1 &= 3-2r-s \\ x_2 &= r \\ x_3 &= -1+s \\ x_4 &= s. \end{aligned}$$

We can write this solution in vector form as

$$(x_1,x_2,x_3,x_4)=(3-2r-s,r,-1+s,s)=(3,0,-1,0)+r(-2,1,0,0)+s((-1,0,1,1).$$

Does this have a geometric interpretation? Yes! It represents the plane in four space that goes through the point (3,0,-1,0) and is parallel to the two vectors (-2,1,0,0) and (-1,0,1,1). ■

We have now seen that a system of linear equations may have no solutions, exactly one solution, or infinitely many solutions. Further we have seen how to find which of these three possibilities holds and how to find all solutions by row reduction for any number of equations in any number of unknowns.

Concepts Review

1. A(n) ____________ to a system of equations is an assignment of numbers to the variables that makes each equation a true statement.
2. Two systems of equations are ____________ if they have exactly the same solutions.
3. A row multiple ____________ ____________ ____________ consists of multiplying a row by a number.
4. A row sum ____________ ____________ ____________ consists of ____________ ____________ ____________ ____________ one row to another.
5. A row interchange ____________ ____________ ____________ consists of ____________ two rows.
6. The first nonzero entry in a row is called ____________ .
7. A matrix is ____________ ____________ if each column containing a pivotal entry has all its other entries equal to 0 and if all pivotal entries are ____________ .
8. When we choose a pivotal entry in row reduction, that entry is called the ____________ .
9. A variable x_i is called ____________ if column i of the row reduced augmented matrix contains a pivotal entry.
10. A parameter is a symbol which stands for a ____________ ____________ .

A. Exercises

In Exercises 1–6, rewrite the system of equations given as a matrix equation.

1. $$\begin{aligned} x_1+x_2 &= 4 \\ x_1-2x_2 &= 1 \end{aligned}$$

2. $$\begin{aligned} 2x_1-3x_2 &= 2 \\ x_1+x_2 &= 6 \end{aligned}$$

3. $$\begin{aligned} x_1-x_2+x_3 &= 1 \\ 3x_1+2x_2-4x_3 &= 1 \\ x_1+2x_2+x_3 &= 4 \end{aligned}$$

4. $$\begin{aligned} x_1-2x_2+3x_3 &= 2 \\ x_1-x_2-x_3 &= 2 \\ 3x_1-2x_2-3x_3 &= 2 \end{aligned}$$

$$5.\quad \begin{array}{rcrcrcr} x_1 & & & + & x_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 2 \\ & & x_2 & + & 3x_3 & = & 3 \end{array}$$

(*Hint*: Write as $x_1 + 0x_2 + x_3 = 2$)

$$6.\quad \begin{array}{rcrcrcr} 5x_1 & + & x_2 & & -4x_3 & = & 2 \\ x_1 & & & & -x_3 & = & 0 \\ & & x_2 & + & 2x_3 & = & 3 \end{array}$$

In Exercises 7–12, use the pure elimination method to solve the system of equations in the exercise specified and show the matrix equation which corresponds to each stage of the solution.

7. Exercise 1
8. Exercise 2
9. Exercise 3
10. Exercise 4
11. Exercise 5
12. Exercise 6

B. Exercises

13. Circle the pivotal entries in each of the following matrices:

(a) $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix}$

14. Circle the pivotal entries in each of the following matrices:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 4 & -3 \\ -3 & 2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

15. Which of the matrices in Exercise 13 is row reduced?

16. Which of the matrices in Exercise 14 is row reduced?

In Exercises 17–22 row reduce the matrix given.

17. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{bmatrix}$
18. $\begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 2 & 4 & 6 \end{bmatrix}$
20. $\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 3 & 3 & 1 \\ -1 & 2 & -3 & 2 \\ 4 & 10 & 6 & -2 \end{bmatrix}$
22. $\begin{bmatrix} 2 & -1 & 6 & 4 \\ 4 & -2 & 4 & 2 \\ -6 & 3 & 6 & 4 \end{bmatrix}$

C. Exercises

23. Rewrite each system of equations below as a matrix equation.

(a) $\begin{array}{rcl} 2x_1 + 3x_2 - x_3 & = & 4 \\ 4x_1 + 5x_2 - 4x_3 & = & 5 \\ x_1 + 2x_2 + x_3 & = & 4 \end{array}$ (b) $\begin{array}{rcl} x_1 + 2x_2 - x_3 & = & 2 \\ 2x_1 + 3x_2 + x_3 & = & 1 \\ -x_1 + x_2 + x_3 & = & -2 \end{array}$

(c) $\begin{array}{rcl} \frac{1}{2}x_1 + x_2 - x_3 & = & 2 \\ 3x_1 + 3x_2 - x_3 & = & 5 \\ 2x_1 + 2x_2 - x_3 & = & 7 \end{array}$ (d) $\begin{array}{rcl} x_1 + 2x_2 + 4x_3 & = & 0 \\ \frac{1}{2}x_1 + x_2 - x_3 & = & 3 \\ x_1 + x_2 + x_3 & = & 2 \end{array}$

24. Rewrite each system of equations below as a matrix equation.

(a) $$\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y + z &= 2 \\ x + y + \tfrac{1}{2}z &= 0 \\ \tfrac{1}{4}x + y + \tfrac{1}{4}z &= -2 \end{aligned}$$

(b) $$\begin{aligned} -x + y - 2z &= -1 \\ x + y - z &= 0 \\ 2x - 3y + z &= -1 \end{aligned}$$

(c) $$\begin{aligned} 3x_1 - 3x_2 + 6x_3 &= 3 \\ 2x_1 + x_2 - 3x_3 &= -1 \\ -x_1 + 4x_2 - 2x_3 &= 5 \end{aligned}$$

(d) $$\begin{aligned} 3x_1 + x_2 - 3x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 2 \\ -3x_1 - x_2 + 2x_3 &= -2 \end{aligned}$$

25. Solve each system of equations in Exercise 23 by row reduction of an augmented matrix.

26. Solve each system of equations in Exercise 24 by row reduction of an augmented matrix.

27. For each augmented matrix below, state whether the associated system of equations has no solutions, one solution, or infinitely many solutions.

(a) $$\left[\begin{array}{cccr|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

(b) $$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(c) $$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \end{array}\right]$$

(d) $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

28. For each augmented matrix below, state whether the associated system of equations has no solutions, one solution, or infinitely many solutions.

(a) $$\left[\begin{array}{ccccr|c} 1 & 0 & 1 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) $$\left[\begin{array}{ccccr|c} 1 & 0 & 1 & 2 & 3 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(c) $$\left[\begin{array}{cccc|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & \text{-3} \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

(d) $$\left[\begin{array}{ccccr|c} 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

29. Write down the solutions of the systems of equations in Exercise 27 that have solutions.

30. Write down the solutions of the systems of equations in Exercise 28 that have solutions.

In Exercises 31–38, find all solutions to the system of equations given

31. $$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 1 \\ x_1 \quad\quad - x_3 + 2x_4 &= 1 \\ x_2 + x_3 + x_4 &= 3 \\ 2x_1 \quad\quad + x_3 + 2x_4 &= 4 \end{aligned}$$

32. $$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 0 \\ x_1 - x_2 \quad\quad + 3x_4 &= 7 \\ 2x_1 - x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 2x_2 + 3x_3 - 3x_4 &= 4 \end{aligned}$$

33. $$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 1 \\ x_1 \quad\quad - x_3 + 2x_4 &= 1 \\ x_2 + x_3 + x_4 &= 3 \\ 2x_1 \quad\quad + x_3 + 2x_4 &= 5 \end{aligned}$$

34. $$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 0 \\ x_1 - x_2 \quad\quad + 3x_4 &= 7 \\ 2x_1 - x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 2x_2 + 3x_3 - 3x_4 &= -1 \end{aligned}$$

35. $$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 + \quad\quad x_4 &= 0 \\ 2x_1 + x_2 - x_3 \quad\quad &= 0 \end{aligned}$$

36. $$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 0 \\ 3x_1 + x_2 - x_3 - 2x_4 &= 0 \\ 2x_1 - x_2 \quad\quad - x_4 &= 0 \end{aligned}$$

37. $$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 2 \\ x_1 + x_2 \quad\quad + x_4 &= 2 \\ 2x_1 + x_2 - x_3 \quad\quad &= 2 \end{aligned}$$

38. $$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 1 \\ 3x_1 + x_2 - x_3 - 2x_4 &= 1 \\ 2x_1 - x_2 \quad\quad - x_4 &= 0 \end{aligned}$$

Problems

1. Milk is sold in quart and half-gallon boxes and in gallon jugs. A grocer pays 50 cents for a quart, 80 cents for a half gallon, and \$1.40 for a gallon of milk.
   (a) Write an equation which says that the grocer buys x_1 quarts, x_2 half gallons, and x_3 gallons of milk for \$100.
   (b) A quart of milk takes up 10 inches of shelf space, a half gallon takes up 16 square inches, and a gallon takes up 36 square inches. Write an equation which says the grocer has (and uses) 2800 square inches of shelf space for the milk.
   (c) Find all solutions to the equations in (a) and (b).
   (d) The grocer has observed that the numbers of half gallons and gallons sold are about equal. Write this as an equation.
   (e) Find all solutions to the equations in (a), (b), and (d).
   (f) Write an equation saying the number of quarts ordered is half the number of half gallons.
   (g) Can the grocer make an order satisfying the conditions of equations (a), (b), (d), and (f)?
2. A company makes white, whole wheat, rye, and mixed grain bread. White bread uses only white flour, whole wheat uses $\frac{1}{3}$ white flour and $\frac{2}{3}$ whole wheat flour, rye bread uses $\frac{1}{2}$ white flour and $\frac{1}{2}$ rye flour, and mixed grain bread uses $\frac{1}{3}$ of each kind of flour. Assuming each loaf of bread uses $\frac{1}{2}$ pound of flour, write down a formula for the amount of white flour in x_1 loaves of white bread, x_2 loaves of whole wheat, x_3 loaves of rye bread, and x_4 loaves of mixed grain bread. Write formulas for the amount of rye flour and whole wheat flour used. Find all combinations of bread the company can produce using 350 pounds of white flour, 150 pounds of whole wheat flour, and 100 pounds of rye flour.
3. Which of the three possibilities (exactly one solution, infinitely many solutions, no solutions) can occur with three equations in four unknowns? What about with four equations in three unknowns?

4. Write a computer program that accepts a matrix and row reduces it to a row reduced matrix.
5. Which of the three possibilities (exactly one solution, infinitely many solutions, no solutions) can occur with a homogeneous system of equations, a system whose matrix equation has the form $MX = 0$? If there is a unique solution, what must it be?
6. Show that if M is a square matrix and when $M|B$ is reduced to $M'|B'$ with M' in row-reduced form, all variables are pivotal, then the rows of M' are the rows of I, perhaps in a different order.
7. Which of the three possibilities (exactly one solution, infinitely many solutions, no solutions) can occur when there are more unknowns than equations? More equations than unknowns?
8. We have seen that if we row reduce $M|B$ to $M'|B'$ and M' happens to be the identity, then B' is the solution matrix for $MX = B$. A row reduced matrix is said to be in *echelon form* if each pivotal entry occurs to the right of all pivotal entries in higher rows. Explain how you could follow the row reduction algorithm with a sorting algorithm such as selection sort, properly modified, to row reduce a matrix to row reduced echelon form.
9. Explain why a square matrix can be row reduced either to an identity matrix or a matrix with a row of zeros.
10. Prove that if $MX = B$ has a unique solution for a given column matrix B, then $MX = C$ has a unique solution for every column matrix C.
11. Row reduced echelon form is described in Problem 8. Prove that for each matrix there is one and only one row reduced echelon matrix to which it may be reduced.

Section 1-3
Inverse and Elementary Matrices

A Elementary Matrices

The matrix equation $MX = B$ looks quite similar to the numerical equation $mx = b$. We solve this numerical equation by multiplying by $m^{-1} = \frac{1}{m}$ to get $x = m^{-1}b$. There is a similar way to solve some matrix equations of the form $MX = B$. The matrix M in the equation $MX = B$ has to be a special kind of square matrix called an invertible matrix for the same kind of technique to work.

M is called **invertible** if there is a matrix M^{-1} such that
$$M^{-1}M = MM^{-1} = I.$$

The matrix M^{-1} is called the **inverse** of M.

EXAMPLE 24 Show that the matrix

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M$$

has an inverse.

Solution We want a matrix M^{-1} such that $MM^{-1} = M^{-1}M = I$. We have no special way to find M^{-1}, so we try guessing. A logical guess for M^{-1} is

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}$$

To check that we have guessed correctly, we write

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot \frac{1}{3} + 0 \cdot 0 & 3 \cdot 0 + 0 \cdot 1 \\ 0 \cdot \frac{1}{3} + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot \frac{1}{3} + 0 \cdot 0 & 3 \cdot 0 + 0 \cdot 1 \\ 0 \cdot \frac{1}{3} + 1 \cdot 0 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■

It will not always be so easy to guess the inverse matrix.

EXAMPLE 25 Show that the matrices

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

are not inverses to each other.

Solution We multiply the two matrices to get

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 + \frac{3}{4} & \frac{2}{3} + \frac{3}{2} \\ 2 + \frac{1}{2} & \frac{4}{3} + 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore the product of the matrices is not I, so they are not inverses. ■

EXAMPLE 26 Show that the matrices below are inverses to each other,

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Solution We multiply the two matrices in both orders to get

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2} & \frac{6}{8} - \frac{3}{4} \\ -\frac{4}{4} + \frac{2}{2} & \frac{12}{8} - \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{4} + \frac{12}{8} & -\frac{3}{4} + \frac{6}{8} \\ \frac{3}{2} - \frac{4}{4} & \frac{3}{2} - \frac{2}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

■

Examples 25 and 26 indicate that guessing is unlikely to be a useful technique for finding inverses of matrices.

We were able to guess the inverse of the matrix M in Example 24 because it has a simple form. Namely, we can obtain M from the identity by performing the row operation of multiplying row 1 by 3, so it is not surprising that we get its inverse from the identity by performing the row operation of multiplying row 1 by $\frac{1}{3}$.

A matrix we get by performing one elementary row operation
on an identity matrix is called an
elementary matrix.

We shall soon see that each elementary matrix has an inverse. It turns out that we can use elementary matrices to develop a technique for finding inverses of all invertible matrices.

Since there are three kinds of elementary row operations, there are 3 types of elementary matrices. The first kind of elementary matrix is a **row multiple matrix**. We use $E(rRi)$ to stand for the result of multiplying row i of I by r. Thus the matrix M in Example 24 is $E(3R1)$ and its inverse is $E(\frac{1}{3}R1)$.

The second kind of elementary matrix is called a **row sum matrix**. We use the notation $E(rR_i + R_j)$ to stand for the result of adding r times row i of I to row j.

EXAMPLE 27 What is the 3-by-3 elementary matrix $E(4R3 + R1)$?

Solution We add four times row 3 of the 3-by-3 identity matrix to row 1 to get

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

The third kind of elementary matrix is called a **row interchange matrix**. We use $E(R_i \leftrightarrow R_j)$ to stand for the result of exchanging row i and row j of an identity matrix.

EXAMPLE 28 What is the 4-by-4 elementary matrix $E(R2 \leftrightarrow R4)$?

Solution We exchange row 2 and row 4 of the 4-by-4 identity matrix to get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

■

As Theorem 5 shows, elementary matrices provide the relationship between matrix multiplication and elementary row operations.

Theorem 5 If E is an elementary matrix and the product EM is defined, then EM is the result of performing on M the elementary row operation used to define E.

Proof Suppose we obtain E by multiplying row i of I by r. Then row i of EM is row i of E times M, which is r times row i of I times M. This is r times row i of M. Any other row of EM is the corresponding row of IM and so is the corresponding row of M. We deal with the other two kinds of elementary matrices similarly. ■

EXAMPLE 29 Using the matrix M given below, compute the product EM for each of the three elementary matrices $E(5R1 + R2)$, $E(5R2)$ and $E(R1 \leftrightarrow R2)$.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution

$$E(5R1 + R2)M = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 15 & 21 \\ 7 & 8 & 9 \end{bmatrix}$$

$$E(5R2)M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 20 & 25 & 30 \\ 7 & 8 & 9 \end{bmatrix}$$

$$E(R1 \leftrightarrow R2)M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

■

Theorem 6 shows that each elementary matrix is invertible.

Theorem 6 Each elementary matrix has an inverse which is also an elementary matrix as follows:

1. The inverse of $E(rRi)$ is $E(\frac{1}{r}Ri)$
2. The inverse of $E(rRi + Rj)$ is $E(-rRi + Rj)$
3. The inverse of $E(Ri \leftrightarrow Rj)$ is $E(Ri \leftrightarrow Rj)$

Proof We leave it to the reader to verify by using Theorem 5 that multiplying the two matrices given in (a) or (b) or (c) in either order gives an identity. ■

EXAMPLE 30 Write down the inverses of the matrices $E(5R1+R2)$, $E(5R2)$, and $E(R1 \leftrightarrow R2)$ of Example 29.

Solution By Theorem 6, we may write

$$E(5R1 + R2)^{-1} = E(-5R1 + R2) = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E(5R2)^{-1} = E(\tfrac{1}{5}R2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$E(R1 \leftrightarrow R2)^{-1} = E(R1 \leftrightarrow R2) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■

B Inverse Matrices

The fact that multiplying M by an elementary matrix is the same thing as performing an elementary row operation on M gives us a way to find inverses for many more matrices.

Theorem 7 Suppose the matrix M can be row reduced to the identity by multiplying by the elementary matrices $E_1, E_2, \ldots, E_n$. Then

1. $M = E_1^{-1}E_2^{-1}\ldots E_n^{-1}$
2. M has an inverse
3. $M^{-1} = E_nE_{n-1}\ldots E_1$

Proof We prove the case $n = 2$; mathematical induction may be used to extend the proof to larger n.

For $n = 2$, we are supposing that $E_2E_1M = I$. Multiplying by E_2^{-1} gives $E_1M = E_2^{-1}I = E_2^{-1}$; now multiplying by E_1^{-1} gives $M = E_1^{-1}E_2^{-1}$, proving Statement 1 in the case $n = 2$.

We prove Statements 2 and 3 simultaneously by showing that E_2E_1 is the inverse to M. By using the fact that $M = E_1^{-1}E_2^{-1}$, the associative law, and definition of inverse matrices we may write

$$E_2E_1M = (E_2E_1)(E_1^{-1}E_2^{-1}) = E_2(E_1E_1^{-1})E_2^{-1} = E_2IE_2^{-1} = E_2E_2^{-1} = I$$

and

$$ME_2E_1 = (E_1^{-1}E_2^{-1})E_2E_1 = E_1^{-1}(E_2^{-1}E_2)E_1 = E_1^{-1}IE_1 = E_1^{-1}E_1 = I$$

This proves statements (2) and (3). ■

Theorem 7 has a wide variety of applications. One of them tells us that one way to test a matrix for invertibility is to see if it may be row reduced to the identity.

EXAMPLE 31 Show that the matrix M below has an inverse.

$$M = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 2 \\ 2 & 6 & 12 \end{bmatrix}$$

Solution We use row reduction to write

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 2 \\ 2 & 6 & 12 \end{bmatrix} \xrightarrow{-2R1+R3} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[-\frac{5}{2}R3+R1]{-R3+R2}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{-R2+R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}R3]{\frac{1}{3}R2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since M can be row reduced to the identity, M has an inverse. ■

Theorem 7 also helps us find inverse matrices by telling us how to find elementary matrices we can multiply together to find an inverse matrix.

EXAMPLE 32 Write down a list of elementary matrices whose product is the inverse M^{-1} of the matrix M in Example 31.

Solution The elementary matrices that correspond to the row operations we performed in Example 31 are: $E_1 = E(-2R1 + R3)$, $E_2 = E(-R3 + R2)$, $E^3 = (-\frac{5}{2}R3 + R1)$, $E_4 = E(-R2 + R1)$, $E_5 = E(\frac{1}{3}R2)$, $E_6 = E(\frac{1}{2}R3)$. By Theorem 7, $M^{-1} = E_6E_5E_4E^3E_2E_1$. ■

Finally, Theorem 7 tells us how we can "factor" some matrices into a product of elementary matrices.

EXAMPLE 33 Write the matrix M of Example 31 as a product of elementary matrices.

Solution By Theorem 7, $M = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$, using the matrices E_i of Example 32. Using Theorem 6 to determine these inverses gives us

$$M = E(2R1 + R3)E(R3 + R2)E(\tfrac{2}{5}R3 + R1)E(R2 + R1)E(3R2)E(2R3)$$

■

If the instructions in Example 32 or 33 had told us to compute M^{-1} rather than give the list of matrices, we would have had considerable work to do. Fortunately, there is a way to bypass this work. Just as we solve the equation $MX = B$ for X by row reducing the augmented matrix $M|B$, we solve the equation $MM^{-1} = I$ for M^{-1} by row reducing the augmented matrix $M|I$.

Theorem 8 If M is a matrix which may be row reduced to the identity, then forming the augmented matrix $M|I$ (which has the entries of the m-by-m matrix M to the left of the line and the entries of the m-by-m identity matrix to the right of the line), and row reducing until the side to the left of the line is the identity gives the augmented matrix $I|M^{-1}$ with M^{-1} to the right-hand side of the line.

Proof Suppose $E_1, E_2, \ldots, E_n$ are the elementary matrices corresponding to the operations used to reduce M to the identity. Then performing these operations on the augmented matrix gives

$$\begin{aligned}(E_n \ldots E_2E_1)(M|I) &= (E_m \ldots E_2E_1M)|(E_m \ldots E_2E_1I)\\ &= M^{-1}M|M^{-1}I\\ &= I|M^{-1}\end{aligned}$$

■

EXAMPLE 34 Find an inverse for the matrix

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} = N$$

Solution We proceed as below to row reduce the augmented matrix.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-2R1+R2} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{array}\right]$$

$$\underset{-\frac{1}{4}R2}{\overset{\frac{1}{2}R1}{\longrightarrow}} \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right] \xrightarrow{\left(-\frac{3}{2}\right)R2+R1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{2}{8} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{4} \end{array}\right]$$

Thus by Theorem 7, N has an inverse, and by Theorem 8,

$$N^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

■

Unique Solutions and Invertible Matrices

We introduced the idea of inverse matrices in a discussion of solving a matrix equation $MX = B$ as we would solve a numerical equation $mx = b$. Theorem 9 tells us that when M is invertible, the analogy works perfectly.

Theorem 9 If an $n \times n$ matrix M has an inverse, then for each n entry column matrix B, the matrix equation $MX = B$ has one and only one solution, $M^{-1}B$.

Proof Multiplying $MX = B$ on the left by M^{-1} gives $X = M^{-1}B$. However, this completely specifies X as a column matrix of constants. By substitution, we see that $X = M^{-1}B$ satisfies the equation $MX = B$. Thus $X = M^{-1}B$ is the one and only one solution to $MX = B$. ■

EXAMPLE 35 Use the inverse matrix you computed in Example 34 to solve the system of equations

$$\begin{aligned} 2x + 3y &= 4 \\ 4x + 2y &= 8 \end{aligned}$$

Solution This system of equations may be written as

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [\,48\,]$$

Multiplying by N^{-1} gives

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 & +3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Thus the unique solution to our system of equations is $x = 2$, $y = 0$. ■

We have learned that if a matrix can be row reduced to the identity then it has an inverse; further, we have an algorithm to carry out in order to find that inverse. We don't yet know whether a square matrix that cannot be row reduced to an identity matrix might still be able to have an inverse. Theorem 10 tells us that this cannot happen.

Theorem 10 If a matrix M has an inverse then it may be row reduced to the identity.

Proof If M has an inverse, then the system of equations $MX = B$ has a unique solution for each column vector B. From Theorem 4, we know that this means each variable must be pivotal in the system of equations corresponding to the augmented matrix $M'|B'$ we get from row reducing $M|B$. As in Problem 6 of Section 10–3, we may conclude, from the facts that M' is square and all variables are pivotal, that the rows of M' are the rows of the identity matrix but perhaps in a different order. Thus M can be row reduced to the identity. ■

Theorem 7 and Theorem 10 let us reach the following conclusion.

A matrix has an inverse if and only if it may be row reduced to the identity.

EXAMPLE 36 Is the matrix $M = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 0 & 5 & -1 \end{bmatrix}$ invertible?

Solution We row reduce M as follows.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -2 \\ 0 & 5 & -1 \end{bmatrix} \xrightarrow{-3R1+R2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 1 \\ 0 & 5 & -1 \end{bmatrix} \xrightarrow{R2+R3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since M may be row reduced to a matrix with a row of zeros, M may not be row reduced to the identity. Therefore M is not invertible. ■

Concepts Review

1. A matrix M is called __________ if there is a matrix N such that $MN = NM = I$.
2. If MX is defined and M' is the result of performing a certain row operation on M, then __________ is the result of performing that operation on MX.
3. A matrix has an inverse if and only if it can be __________ __________ to the identity.
4. A matrix obtained by performing an elementary row operation on an identity matrix is called an __________ matrix.
5. Multiplying M on the left by an elementary matrix E has the same effect as performing the __________ __________ __________ defining E on M.
6. The inverse of an elementary matrix of a given type is a(n) __________ __________ of the __________ type.
7. A product of invertible matrices is __________ .
8. If M is invertible, then the matrix equation $MX = B$ has __________ __________ __________ __________ solution(s).

A. Exercises

In Exercises 1–12, write down the 3-by-3 elementary matrix specified.

1. $E(2R1)$	2. $E(R2 \leftrightarrow R3)$	3. $E(-R1 + R2)$
4. $E(2R3 + R1)$	5. $E(R1 \leftrightarrow R3)$	6. $E(2R3)$
7. $E(-3R2)$	8. $E(R1 \leftrightarrow R2)$	9. $E(-2R1 + R3)$
10. $E(4R1)$	11. $E(R1 \leftrightarrow R1)$	12. $E(3R2 + R1)$

In Exercises 13–24, show the result of writing each of the matrices given on the left of the matrix M below and performing the indicated multiplication.

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

13. $E(2R1)$	14. $E(R2 \leftrightarrow R3)$	15. $E(-R1 + R2)$
16. $E(2R3 + R1)$	17. $E(R1 \leftrightarrow R3)$	18. $E(2R3)$
19. $E(-3R2)$	20. $E(R1 \leftrightarrow R2)$	21. $E(-2R1 + R3)$
22. $E(4R1)$	23. $E(R1 \leftrightarrow R1)$	24. $E(3R2 + R1)$

In Exercises 25–36, write down the inverse of the 3-by-3 elementary matrix specified.

25. $E(2R1)$	26. $E(R2 \leftrightarrow R3)$	27. $E(-R1 + R2)$
28. $E(2R3 + R1)$	29. $E(R1 \leftrightarrow R3)$	30. $E(2R3)$
31. $E(-3R2)$	32. $E(R1 \leftrightarrow R2)$	33. $E(-2R1 + R3)$
34. $E(4R1)$	35. $E(R1 \leftrightarrow R1)$	36. $E(3R2 + R1)$

In Exercises 37–40, verify that the pair of matrices given is a pair of inverse matrices.

37. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} -\frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$

38. $\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{4} \\ -1 & 0 & \frac{1}{2} \end{bmatrix}$

39. $\begin{bmatrix} -1 & -1 & -1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

40. $\begin{bmatrix} -4 & 2 & 1 \\ -5 & 2 & 1 \\ -3 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -1 \\ 1 & -2 & 2 \end{bmatrix}$

B. Exercises

41. Show that the matrix M below is invertible by using row reduction.

$$M = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

42. Show that the matrix N below is invertible by using row reduction.

$$N = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ 1 & 1 & 2 \end{bmatrix}$$

43. Use the row operations from Exercise 41 to write down a sequence $E_1, E_2, \ldots, E_n$ of matrices such that
$$E_n \ldots E_4 E_3 E_2 E_1 M = I$$
How does the product $E_n E_{n-1} \ldots E_2 E_1$ relate to M?

44. Use the row operations from Exercise 42 to write a sequence $E_1, E_2, E_3, \ldots, E_n$ of matrices such that
$$E_n E_{n-1} \ldots E_2 E_1 M = I.$$
How does the product $E_n E_{n-1} \ldots E_2 E_1$ relate to M?

45. The results of Exercises 41 and 43 allow you to write M as a product of elementary matrices. What are the matrices ? In what order must they be multiplied ?

46. The results of Exercises 42 and 44 allow you to write N as a product of elementary matrices. What are the matrices? In what order must they be multiplied ?

In Exercises 47–54, find the inverse of each of the matrices given.

47. $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$ 48. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

49. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ 50. $\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$

51. $\begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}$ 52. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

53. $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = M$ 54. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix} = N$

55. Use the matrix you computed in Exercise 47 to solve the equations

$$\begin{aligned} -x + 3y &= 2 \\ 2x - y &= 3 \end{aligned}$$

56. Use the matrix you computed in Exercise 50 to solve the equations

$$\begin{aligned} x + 2y &= 3 \\ x + 4y &= 1 \end{aligned}$$

57. Using the matrix you computed in Exercise 53, solve the system of equations

$$MX = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

58. Using the matrix you computed in Exercise 54, solve the system of equations

$$NX = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

59. Determine whether each matrix below is invertible.

(a) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 2 & 3 \\ 1 & -1 & 4 & 1 \\ 3 & 6 & -3 & 1 \\ 3 & 2 & -1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}$

60. Determine whether each matrix below is invertible.

(a) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & -2 \\ 2 & 1 & 1 \\ 1 & -3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

61. For each invertible matrix in Exercise 59, find its inverse.

62. For each invertible matrix in Exercise 60, find its inverse.

63. For each matrix M in Exercise 59, the matrix equation $MX = 0$, where X is a column vector of variables and 0 is a column vector of zeros, has either exactly one solution or infinitely many solutions. For each possible M, which of these is the case?

64. For each possible matrix M in Exercise 60, the matrix equation $MX = 0$, where X is a column vector of variables and 0 is a column vector of zeros, has either exactly one solution or infinitely many solutions. For each M in Exercise 60, which of these is the case?

65. Let B stand for the all-ones column vector. For each invertible matrix M in Exercise 59, use the inverse you computed in Exercise 61 to solve the system of equations $MX = B$.

66. Let B stand for the all ones column matrix. For each invertible matrix M in Exercise 60, use the inverse you computed in Exercise 62 to solve the system of equations $MX = B$.

67. For each noninvertible matrix M in Exercise 59, determine all solutions to the matrix equation $MX = B$ with B the all-ones column vector as in Exercise 65.

68. For each noninvertible matrix M in Exercise 60, determine all solutions to the matrix equation $MX = B$ where X and B are as in Exercise 66.

Problems

1. Prove by induction that the inverse of a product of n invertible matrices is the product of their inverses in reverse order.
2. Write out the proof that if M is a product of elementary matrices, then M may be row reduced to an identity matrix.
3. Define an elementary column matrix. Show that each elementary row matrix is also an elementary column matrix.
4. Experiment with the result of multiplying the matrices of Exercises 1–12 on the right times the matrix M of Exercises 13–24. What is the effect on a matrix M of multiplying M on the right by an elementary matrix?
5. Is it possible for more than one matrix N to serve as an inverse to a matrix M? Give an example or explain why not.
6. A *right inverse* to a matrix is a matrix N such that $MN = I$. (*Note*: A nonsquare matrix may have a right inverse.)

   (a) Is it possible for a nonsquare matrix to have more than one right inverse? (Give examples or explain why not.)

(b) Is it possible for a square matrix to have more than one right inverse? (Give examples or explain why not.)

7. Prove or give a counter-example: If a square matrix has a right inverse, then it is invertible (right inverse was defined in Problem 6.)
8. In the proof of Theorem 5, show how to deal with the other two kinds of elementary row operations.
9. What is the smallest number n such that every invertible two-by-two matrix is a product of n or fewer elementary matrices?
10. An *upper triangular* matrix M has the property that $M_{ij} = 0$ if $i > j$.

    (a) Write down a typical 3-by-3 upper triangular matrix.

    (b) Explain why a 3-by-3 upper triangular matrix is invertible if and only if the product of the entries along its main diagonal is not zero.

    (c) What is the smallest n such that every invertible 3-by-3 upper triangular matrix factors into n or fewer elementary matrices?

11. Extend the proof of Theorem 7 to the product of n elementary matrices.
12. If M is the adjacency matrix of a graph with n vertices and $I - M$ has an inverse, how does $(I - M^n)n(I - M)^{-1}$ relate to the transitive closure of the graph?
13. Prove or give a counter-example: If M is the adjacency matrix of a connected graph, then $I - M$ is invertible.

Section 1-4 The Definition of Determinants

A The Determinant Axioms

The ordinary (numerical) equation $mx = b$ in which m and b are numbers and x is a variable has a unique solution if and only if $m \neq 0$. The matrix equation $MX = B$ has a unique solution if and only if M is invertible; it would be convenient if there were one number m related to M which would determine whether or not M is invertible according to whether or not that number is different from 0. You may have seen such a number for 2-by-2 matrices; people frequently are taught about *determinants* in connection with two equations in two unknowns.

Determinants of Two-by-Two Matrices

The matrix equation

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \text{or} \qquad AX = B$$

has a unique solution if and only if

$$a_{11}a_{22} - a_{12}a_{21} \neq 0$$

The number $a_{11}a_{22} - a_{12}a_{21}$ is called the **determinant** of the 2-by-2 matrix A and denoted by **det** (**A**).

To see how it arises, consider the augmented matrix

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Multiplying row 2 by a_{11} and subtracting a_{21} times row 1 from it gives

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a_{22}a_{11} - a_{12}a_{21} & a_{11}b_2 - a_{21}b_1 \end{array}\right]$$

If $a_{22}a_{11} - a_{12}a_{21} \neq 0$, we may divide by it and complete the row reduction to get

$$\left[\begin{array}{cc|c} 1 & 0 & \dfrac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}} \\ & & \\ 0 & 1 & \dfrac{a_{11}b_2 - a_{21}b_1}{a_{22}a_{11} - a_{12}a_{21}} \end{array}\right]$$

Technically, we were assuming that a_{11} was nonzero (otherwise we would have been multiplying row 2 of the augmented matrix by zero). It is possible to work out a row reduction to get the same formula for x_1 and x_2 that we get from this row reduction in any case. Looking back, you see that if $a_{22}a_{11} - a_{12}a_{21}$ *is* zero, we get a row of zeros for our second row to the left of the line in our row reduction, so we have either no solution or infinitely many solutions. Thus $AX = 0$ has a unique solution if and only if the determinant of the coefficient matrix is nonzero. Therefore a 2-by-2 matrix is invertible if and only if its determinant is nonzero.

There are similar determinants associated with 3-by-3, 4-by-4, and larger matrices; however, trying to discover formulas for them by finding general formulas for solutions to more equations in more unknowns would involve us in considerable algebra.

Rules for Computing Determinants

Fortunately, we can avoid a great deal of algebra by defining a determinant of a matrix through a set of rules (called the **determinant axioms**) for computing determinants. We will figure out what these rules should be by analyzing how the determinants of two-by-two matrices interact with the row operations we have used for inverting matrices. Our first type of row operation consists of multiplying a row by a number. Since

$$\begin{aligned} \det \begin{bmatrix} ra_{11} & ra_{12} \\ a_{21} & a_{22} \end{bmatrix} &= ra_{11}a_{22} - ra_{12}a_{21} \\ &= r(a_{11}a_{22} - a_{12}a_{21}) = r \det \begin{bmatrix} ra_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{aligned}$$

it is natural to write our first rule as follows.

Multiplying a Row by a Real Number

Determinant Rule (1), the Row Multiple Rule:
Multiplying row i of a matrix M by the real number r multiplies the determinant of M by r.

In symbols, we would write

$$\textit{Determinant rule (1)}: \quad \det \begin{bmatrix} R1 \\ \vdots \\ rRi \\ \vdots \\ Rn \end{bmatrix} = r \det \begin{bmatrix} R1 \\ \vdots \\ Ri \\ \vdots \\ Rn \end{bmatrix}$$

Another way to state Rule (1) is as follows:

If M' is obtained from M by factoring r from row i of M, then $\det M = r \det M'$.

EXAMPLE 37 Find what happens to the determinant of the matrix

$$\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$$

if you multiply row 1 by 2.

Solution We know that

$$\det \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} = 9 - 4 = 5$$

and

$$\det \begin{bmatrix} 2 \cdot 3 & 2 \cdot 1 \\ 4 & 3 \end{bmatrix} = \det \begin{bmatrix} 6 & 2 \\ 4 & 3 \end{bmatrix} = 18 - 8 = 10 = 2 \cdot 5$$

■

Adding a Row to Another Row

Our second kind of elementary row operation consists of adding a multiple of one row to another. By applying the definition of a determinant of a two-by-two matrix, we may write

$$\begin{aligned} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + ra_{11} & a_{22} + ra_{12} \end{bmatrix} &= a_{11}(a_{22} + ra_{12}) - (a_{21} + ra_{11})a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12} + a_{11}ra_{12} - ra_{11}a_{12} \\ &= a_{11}a_{22} - a_{21}a_{12} \end{aligned}$$

Thus the determinant didn't change! This suggests the second rule.

Determinant rule (2), the Row Sum Rule:
Adding a numerical multiple of row i of the matrix M to row j of the matrix M doesn't change the determinant of M (if $i \neq j$).

$$\textit{Determinant rule (2)}: \det \begin{bmatrix} \text{—}R1\text{—} \\ \vdots \\ \text{—}Ri\text{—} \\ \vdots \\ \text{—}Rj + rRi\text{—} \\ \vdots \\ \text{—}Rn\text{—} \end{bmatrix} = -\det \begin{bmatrix} \text{—}R1\text{—} \\ \vdots \\ \text{—}Ri\text{—} \\ \vdots \\ \text{—}Rj\text{—} \\ \vdots \\ \text{—}Rn\text{—} \end{bmatrix}$$

EXAMPLE 38 Show the effect on the determinant of adding twice row 2 to row 1 with the matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

Solution We know that $\det \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = 2 \cdot (-1) - 3 \cdot 1 = -5$

and $\det \begin{bmatrix} 2+2 & 3-2 \\ 1 & -1 \end{bmatrix} = \det \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} = 4(-1) - 1 \cdot 1 = -5$ ■

Interchanging Two Rows

Finally, if we experiment with interchanging two rows of a 2-by-2 matrix, we get

$$\det \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} = a_{21}a_{12} - a_{22}a_{11} = -(a_{11}a_{22} - a_{21}a_{12}) = -\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

This suggests the general rule for determinants:

Determinant rule (3), the Row Interchange Rule:
Interchanging two rows of a matrix multiplies its determinant by -1.

In symbols, we can write

$$\textit{Determinant rule (3)}: \det \begin{bmatrix} \text{—}R1\text{—} \\ \vdots \\ \text{—}Ri\text{—} \\ \vdots \\ \text{—}Rj\text{—} \\ \vdots \\ \text{—}Rn\text{—} \end{bmatrix} = -\det \begin{bmatrix} \text{—}R1\text{—} \\ \vdots \\ \text{—}Ri\text{—} \\ \vdots \\ \text{—}Rj\text{—} \\ \vdots \\ \text{—}Rn\text{—} \end{bmatrix}$$

EXAMPLE 39 Show the effect on the determinant of interchanging the rows of

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Solution We write $\det\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 1 \cdot 3 - 2 \cdot 2 = -1$
and $\det\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = 2 \cdot 2 - 3 \cdot 1 = 1$ ■

B Using the Rules to Compute Determinants

Our next two examples show that the rules we have introduced so far are a considerable aid in computing determinants but are not quite sufficient.

A Row of Zeros Makes the Determinant Zero

EXAMPLE 40 Apply the rules to show what information we may learn about the determinant of

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

by using row operations.

Solution Since multiplying row 2 of the matrix M by zero doesn't change it, applying determinant rule (1) gives us

$$\det M = \det \begin{bmatrix} 1 & 3 & 2 \\ 0 \cdot 0 & 0 \cdot 0 & 0 \cdot 0 \\ 1 & 0 & -1 \end{bmatrix} = 0 \cdot \det \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} = 0$$

■

EXAMPLE 41 Apply the rules to show what information we may learn about the determinant of

$$N = \begin{bmatrix} 1 & 6 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

by using row operations.

Solution Subtracting 3 times the second row of N from the first row gives by determinant rule (2),

$$\det \begin{bmatrix} 1 & 6 & 5 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 8 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Now subtracting twice row 3 from row 1 and adding a fourth of row 3 to row 2 in the last matrix we get

$$\det \begin{bmatrix} 1 & 0 & 8 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Now applying rule (1) twice gives

$$\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 2 \cdot r \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We don't know the determinant of a 3-by-3 identity matrix, so this is all we can say so far. ■

The Determinant of an Identity Matrix

In discussing equations, we saw that a matrix can either be row reduced to a matrix with a row of zeros or else can be row reduced to an identity matrix. Thus the example suggests that, once we know what the determinant of an n-by-n identity matrix is, our rules should be complete enough to let us compute any determinant whatsoever. Since the determinant of a 2-by-2 identity matrix is 1, the most natural rule for determining the determinant of an n-by-n identity matrix is

Determinant Rule (4), the Identity Matrix Rule:
The determinant of an identity matrix is 1.

In symbols, we write

$$\textit{Determinant rule (4)}\colon \det I = 1$$

EXAMPLE 42 Complete the computation of $\det N$ in Example 42.

Solution By following the chain of equalities in Example 42, we see that $det(N) = 2 \cdot 4 \det I = 2 \cdot 4 \cdot 1 = 8$. ■

The Definition of the Determinant Function

A function defined on n-by-n matrices is called a **determinant function** if it satisfies determinant rules (1) through (4).

We have seen by examples above how we can use the rules to compute determinants. The method we used in the examples may be summarized. as in Theorem 14.

Theorem 11 If there is a determinant function defined on n-by-n matrices and satisfying rules (1) through (4), then it may be computed by the following procedure.
Row reduce the matrix until you get a row of zeros or an identity matrix. If you get a row of zeros, the determinant is zero. Otherwise the determinant is the product of the following **elementary factors**. There is an elementary factor of r for each row sum operation in which you factor an r out of a row. There is an elementary factor of 1 for each type 2 row operation and an elementary factor of -1 for each row interchange operation, interchanging two rows.

Proof The procedure simply describes how to apply the determinant rules. ■

EXAMPLE 43 Compute the determinant of the matrix

$$\begin{bmatrix} 2 & 8 & 9 \\ -2 & -4 & -3 \\ 2 & 12 & 12 \end{bmatrix}$$

Solution We row reduce the matrix as follows

$$\begin{bmatrix} 2 & 8 & 9 \\ -2 & -4 & -3 \\ 2 & 12 & 12 \end{bmatrix} \xrightarrow[-1R1+R3]{R1+R2} \begin{bmatrix} 2 & 8 & 9 \\ 0 & 4 & 6 \\ 0 & 4 & 3 \end{bmatrix} \xrightarrow{-1R2+R3}$$

$$\begin{bmatrix} 2 & 8 & 9 \\ 0 & 4 & 6 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{-2R2+R1} \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 6 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[2R3+R2]{-1R3+R1}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[\left(-\frac{1}{3}\right)R3]{\substack{\left(\frac{1}{2}\right)R1 \\ \left(\frac{1}{4}\right)R2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

All of the operations except for the last three have elementary factors equal to 1; the elementary factors of the last three operations are, respectively, 2, 4, and -3, so the determinant is $2 \cdot 4 \cdot (-3) = -24$.

Notice how multiplying row 1 by $\frac{1}{2}$ corresponds to factoring out a 2 from row 1 and using rule (1). This shows that multiplication by $\frac{1}{r}$ in row reduction gives an elementary factor of r. The elementary factor of 4 corresponds to factoring a four from row 2 and the elementary factor of -3 corresponds to factoring -3 from row 3. ∎

Note how in both Example 5 and Example 7 when we had a **triangular** matrix, one with zeros below the main diagonal, the determinant we eventually got by row reduction was the product of the diagonal entries. Was this an accident, or does it give us a short cut for computing determinants? Fortunately this was no accident. We call a matrix **upper triangular** if all entries below the main diagonal are zero. With this terminology, we may describe the short cut easily.

Theorem 12 The determinant of an upper triangular matrix is the product of its diagonal entries.

Proof The proof is similar to the computations of Example 5 and the last three steps of Example 45. It is described in Problems 5–7. ∎

EXAMPLE 44 Compute the determinant of

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

Solution By Theorem 15, the determinant is $2 \cdot 3 \cdot (-1) \cdot (-5) = 30$. ■

In Example 45, we started out with a matrix that was not triangular and converted it to a triangular matrix with row sum operations. Thus the determinant of the original matrix and the triangular matrix are the same. This was no accident either; any matrix may be converted to a triangular matrix by using row sum operations. This gives us another way to compute determinants.

Theorem 13 To compute the determinant of a matrix, we row reduce it to triangular form by adding multiples of rows to other rows, then we multiply the diagonal entries together to get the determinant.

Proof All that needs to be proved is that we can row reduce a matrix to triangular form with only row sum operations. Problems 1–4 do this. ■

We already have an example of Theorem 16, namely Example 45. The third matrix in the chain of row reductions is upper triangular and was obtained solely by type 2 operations. Therefore the determinant is $2 \cdot 4 \cdot (-3) = -24$. If you choose to apply Theorem 16, be careful to apply *only* row sum operations in reducing the matrix to triangular form.

Concepts Review

1. The formula for the determinant of a two-by-two matrix A with entries a_{ij} is ____________ .
2. Multiplying row i of a matrix M by the real number r ____________ the determinant of M by ____________ .
3. Adding a numerical multiple of row i to row j of the matrix M ____________ ____________ ____________ the determinant of M.
4. Interchanging two rows of the matrix M ____________ the determinant of M by ____________ .
5. The determinant of an identity matrix is ____________ .
6. There is an ____________ ____________ associated with each row operation on a square matrix, and multiplying them together gives the determinant of the matrix.
7. If all nonzero entries of a square matrix are on or above the main diagonal, then the matrix is called a(n) ____________ ____________ matrix.
8. The product of the ____________ entries is an upper ____________ matrix is equal to the determinant.
9. When we reduce a matrix to triangular form by using row sum elementary row operations there is ____________ effect on the determinant.

A. Exercises

In Exercises 1–8, compute the determinant of the matrix given.

1. $\begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
3. $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$
4. $\begin{bmatrix} -2 & 0 \\ -3 & -2 \end{bmatrix}$
5. $\begin{bmatrix} 0 & 3 \\ -1 & 1 \end{bmatrix}$
6. $\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$
7. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
8. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

In Exercises 9–14, compute the two determinants and explain which determinant rule is illustrated.

9. $\det \begin{bmatrix} 1 & 6 & 2 & 4 \end{bmatrix} \quad \det \begin{bmatrix} 1 & 6 \\ 3 & 10 \end{bmatrix}$
10. $\det \begin{bmatrix} 1 & 6 \\ 2 & 4 \end{bmatrix} \quad \det \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$
11. $\det \begin{bmatrix} 2 & 5 \\ -4 & -8 \end{bmatrix} \quad \det \begin{bmatrix} 2 & 5 \\ 2 & 4 \end{bmatrix}$
12. $\det \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \quad \det \begin{bmatrix} 2 & -6 \\ 2 & -2 \end{bmatrix}$
13. $\det \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \quad \det \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$
14. $\det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} \quad \det \begin{bmatrix} 1 & 5 \\ 1 & -4 \end{bmatrix}$

In Exercises 15–18, the determinants of the pairs of matrices given are related by the determinant rules. For each pair, state how $\det A$ and $\det B$ are related.

15. $A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & -4 & -6 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$
16. $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -4 & -7 \\ 3 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$
17. $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$
18. $A = \begin{bmatrix} 3 & -2 & 5 \\ 2 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

B. Exercises

In Exercises 19–24, find the determinants of the matrices given.

19. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 20. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ 21. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ 23. $\begin{bmatrix} 2 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 24. $\begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

For some matrices in Exercises 25–33, the determinant is the product of diagonal elements, and for some it is not. Compute the determinants and show which ones are and are not the product of the diagonal elements $a_{11}a_{22}a_{33}$ or the reverse diagonal elements $a_{13}a_{22}a_{31}$.

25. $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ 26. $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 6 \\ 3 & 8 & 2 \end{bmatrix}$ 27. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 4 & 2 \end{bmatrix}$

28. $\begin{bmatrix} 2 & 3 & 6 \\ 2 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 29. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ 30. $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 4 & 2 & -3 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 3 & 2 \\ 6 & -1 & 0 \\ 3 & 0 & 0 \end{bmatrix}$ 32. $\begin{bmatrix} 4 & 3 & 2 \\ 0 & 6 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

Compute the determinants of each matrix in Exercises 33–38 by using the method of elementary factors.

33. $\begin{bmatrix} 2 & -3 & 2 \\ 4 & 3 & 4 \\ 2 & -3 & 1 \end{bmatrix}$ 34. $\begin{bmatrix} -3 & 4 & 2 \\ 3 & 4 & 1 \\ -3 & 2 & 1 \end{bmatrix}$ 35. $\begin{bmatrix} -1 & 2 & -3 \\ 0 & 2 & 3 \\ -1 & 4 & -3 \end{bmatrix}$

36. $\begin{bmatrix} 5 & 2 & 6 \\ 5 & 1 & 2 \\ -5 & 1 & 4 \end{bmatrix}$ 37. $\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ 38. $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 3 & 5 \\ 2 & -3 & 1 \end{bmatrix}$

Compute the determinant of each matrix in Exercises 39–44 by row reducing to triangular form.

39. $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ 40. $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 3 & 3 \\ 1 & 1 & 0 \end{bmatrix}$ 41. $\begin{bmatrix} 3 & 3 & -6 \\ 2 & 4 & 0 \\ 1 & 1 & 6 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 4 & -1 \\ 1 & 3 & 3 \end{bmatrix}$ 43. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ 44. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 8 \\ 1 & 1 & 4 \end{bmatrix}$

In Exercises 45–50, compute the determinants by any appropriate method.

45. $\det \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 \\ -1 & -1 & 1 & 1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$ 46. $\det \begin{bmatrix} 0 & 2 & -1 & 0 \\ 1 & 0 & 1 & 3 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \end{bmatrix}$

47. $\det \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 0 & 3 & 4 \\ 2 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

48. $\det \begin{bmatrix} 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \\ 1 & 1 & 16 & 16 \end{bmatrix}$

49. $\det \begin{bmatrix} x & 0 & y & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & -1 & -1 \end{bmatrix}$

50. $\det \begin{bmatrix} x & 1 & 3 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 3 & 2 \\ y & -1 & -1 & 1 \end{bmatrix}$

Problems

1. Describe the effect of the following sequence of row operations on a matrix:

$$R_j + R_i, \quad -1 \cdot R_i + R_j, \quad R_j + R_i, \quad -1 \cdot R_j$$

2. Explain why if a sequence of (nonzero) row multiple and row sum elementary row operations can make the entry M_{ij} of a matrix equal to zero, then a sequence of row sum elementary operations can make M_{ij} equal to zero also.
3. Prove by induction that any n-by-n matrix can be row reduced to an upper triangular matrix.
4. On the basis of Problems 1–3, explain why a matrix may be row reduced to triangular form without changing its determinant.
5. Show that if an upper triangular matrix has zero as a diagonal entry, then by row reduction we can convert the lowest row with a diagaonal zero into an entire row of zeros.
6. Show that if all the diagonal entries of an upper triangular matrix are nonzero, then we may row reduce the matrix to a diagonal matrix without changing its determinant.
7. Using Problems 5 and 6, prove Theorem 15.
8. A more restricted version of determinant rule (2) is

   Rule (2′). Adding row i to row j does not change the determinant.

   Show that rules (1) and (2′) imply rule (2). (*Hint*: Examine how Problem 1 lets you prove that rules (1) and (2) imply rule (3).)
9. A still more restricted version of determinant rule (2) is

   Rule (2″) Adding a row to an adjacent row does not change the determinant.

   Show that rules (1) and (2″) imply rule (2′). (*Hint*: See how to add row i to row $i+2$ by first adding it to $i+1$, then adding row $i+1$ to row $i+2$, and finally doing some subtracting.)
10. On the basis of Problems 1, 8, and 9, show that if a function satisfies determinant rules (1), (2″), and (4), then it satisfies all the determinant rules.
11. Given a two-by-two matrix with rows $R1$ and $R2$, there is a parallelogram whose vertices are the origin, the point whose x- and y-coordinates are given by $R1$, the point whose x- and y-coordinates are given by $R2$, and the point whose x- and y-coordinates are given by $R1 + R2$.

    (a) Explain why the figure is a parallelogram.

    (b) Explain why a row sum elementary matrix operation on M gives a matrix M' whose parallelogram has the same area as M.

(c) Explain why a row multiple elementary row operation with multiple r gives a matrix M' whose parallelogram has an area that is the absolute value of r times the area of the parallelogram of M.

(d) Explain why a row interchange elementary row operation on M gives a matrix M' whose parallelogram has the same area as M.

(e) Explain why the area of the parallelogram determined by M is 0 if M is not invertible and is the product of the avsolute values of the elementary factors of a row reduction of M to the identity if M is invertible.

(f) How does the area of the parallelogram relate to the determinant of M? Could we have used columns to define the parallelogram and gotten the same answer?

(g) Make a believable conjecture about the volume of a parallelpiped (a figure having 8 corners) associated with a three-by-three matrix.

(h) Outline the main points of the proof of your conjecture in (g).

Section 1-5
Properties of Determinants

A Products and Transposes

We introduced determinants in hopes that the determinant of a matrix would be nonzero if and only if the matrix is invertible. We have learned that a matrix is invertible if and only if it is a product of elementary matrices. Now suppose the matrix M is a product of elementary matrices $M = E_1 \cdot E_2 \cdots E_n$. We know that if we first perform the elementary row operation that "undoes" E_1, then the operation that "undoes" $E_2, \ldots$ and finally the one that "undoes" E_n, we will have row reduced M to the identity. We know, from the elementary factors description of determinants, that the determinant of an elementary matrix E is the elementary factor of the row operation that converts E to the identity. Theorem 17 summarizes these remarks.

Determinants of Products of Matrices

Theorem 14 The determinant of a product of elementary matrices is the product of their determinants.

Proof Immediate from the discussion above. ■

Theorem 15 The determinant of a matrix is nonzero if and only if it is invertible.

Proof If a matrix is invertible, then it is a product of elementary matrices and therefore, by Theorem 17 and the fact that each elementary factor is nonzero, its determinant is nonzero. If a matrix is not invertible, then it may be row reduced to a matrix with a row of zeros and therefore has determinant zero. ■

Theorem 17 suggests a more general result which is also true.

Theorem 16 For any two square matrices M and N, $\det MN = \det M \det N$.

Proof If M and N are invertible then Theorem 4 applied to M, N, and MN gives our result. If one of M or N is not invertible, then $\det M \det N$ is zero, so for the formula to hold $\det MN$ would have to be zero—that is, MN would have to be not invertible. In the problems we discuss how to show that if M or N is not invertible, then MN is not invertible. ■

EXAMPLE 45 Find the determinant of the product of the matrices M and N below without actually computing MN.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix} \qquad N = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 5 & 3 \end{bmatrix}$$

Solution Because $\det M = 1 \cdot 4(-1) = -4$, $\det N = 2 \cdot 2 \cdot 3 = 12$, $\det MN = -48$. ■

Elementary Matrices and Column Operations

We have so far concentrated all our thinking on row operations. There are three operations on the columns of matrices, called elementary column operations, defined analogously with elementary row operations. What effect do column operations have on determinants? The elementary matrix formed from I by multiplying row i by r can also be formed by multiplying column i by r. The elementary matrix formed from I by interchanging rows i and j may also be formed by interchanging columns i and j. Adding r times row i of I to row j is equivalent to adding r times column j to column i. Further, it can be shown that multiplying M by an elementary (column) matrix on the right performs the corresponding elementary column operation on M. This gives us the next theorem.

Theorem 17 Performing an elementary column operation on M multiplies the determinant of M by the determinant of the corresponding elementary matrix. That is, it has the same effect as the corresponding elementary row operation.

Proof From the remarks above, this is a consequence of Theorem 19. ■

Transposes Relate Row and Column Operations to Determinants

The matrix N is called the **transpose** of the matrix M is *row* i of N has the same entries in the same order as *column* i of M. We write M^t (read as M-transpose and not as M to the t) for the transpose N of M. People say, "we get M^t from M by interchanging rows and columns."

EXAMPLE 46 Write down the transpose M^t of $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Solution Row 1 should be [1 4 7], row 2 should be [2 5 8], and row 3 should be [3 6 9]. This gives

$$M^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

■

A useful fact about transposes is that $(MN)^t = N^t M^t$.

EXAMPLE 47 Compute M^t, N^t, and $N^t M^t$ and MN for the matrices M and N below.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad N = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Solution We write

$$M^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad N^t = \begin{bmatrix} w & y \\ x & z \end{bmatrix}$$

by definition of transposes. Matrix multiplication gives us

$$MN = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

and

$$N^t M^t = \begin{bmatrix} w & y \\ x & z \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} wa + yb & wc + yd \\ xa + zb & xc + zd \end{bmatrix}$$

This actually proves the formula $(MN)^t = N^t M^t$ for 2-by-2 matrices. ■

Theorem 18 If M is an m-by-m matrix and N is an n-by-k matrix, then $(MN)^t = N^t M^t$.

Proof The i, j entry of $(MN)^t$ is by definition the j, i entry of MN which is the product of row j of M (and thus column j of M^t) with column i of N (and thus row i of N^t). Therefore $(MN)^t$ and $N^t M^t$ have the same entries so they are equal. ■

Theorem 19 For any square matrix M, $\det M = \det M^t$.

Proof If M is invertible, then it is a product of elementary matrices. Since the determinant of an elementary matrix and its transpose must be equal, the theorem follows in this case directly from Theorem 19. We can show that M^t is invertible if and only if M is invertible by using Theorem 21, so $\det M = \det M^t$ if M is not invertible as well. ■

Theorem 22 tells us how to translate a fact involving rows of matrices and determinants into a corresponding fact about columns of matrices and determinants.

B The Additive Property

We described the determinant function originally by stating how we hoped it would behave when we performed elementary row operations. It will be useful to have a property that describes how it reacts to a whole sequence of row operations. This description will lead us to one of the standard formulas for the determinant function and to a proof that there really is a determinant function satisfying axioms (determinant rules) (1)–(4). We shall study the case when we have a whole sequence of row operations all of which affect *only* row i. Thus we might add a numerical multiple of any row, even row i itself, to row i. This replaces row i by

$$Ri + \sum_{j=1}^{n} a_j Rj$$

for some n-tuple of numbers $a_1, a_2, \ldots, a_n$. If our matrix can be row reduced to the identity, then (by row reducing to the identity and then combining rows of the identity together) we can make

$$\sum_{j=1}^{n} a_j Rj$$

into any row matrix R we choose. Thus we should be able to figure out the result on the determinant of adding any row vector we please to row i.

The Effect of Adding an Arbitrary Row Matrix to Row i

Theorem 20 If Ri' is any row matrix and the square matrix M has rows $R1, \ldots Ri, \ldots Rn$, then

$$\det\begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ Ri + Ri' \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix} = \det\begin{bmatrix} R1 \\ \vdots \\ Ri \\ \vdots \\ Rn \end{bmatrix} + \det\begin{bmatrix} R1 \\ \vdots \\ Ri' \\ \vdots \\ Rn \end{bmatrix}$$

Proof If M is invertible, then we may assume that Ri' is

$$\sum_{j=1}^{n} a_j R_j$$

By our second determinant axiom applied once to each row other than row i,

$$\det\begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ Ri + \displaystyle\sum_{j=1}^{n} a_j R_j \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix} = \det\begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ Ri + a_i Ri \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix}$$

Then by determinant axiom (1),

$$\det\begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ Ri + \sum_{j=1}^{n} a_j R_j \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix} = (1+a_i)\det\begin{bmatrix} R1 \\ \vdots \\ Ri \\ \vdots \\ Rn \end{bmatrix} = \det\begin{bmatrix} R1 \\ \vdots \\ Ri \\ \vdots \\ Rn \end{bmatrix}$$

$$+\det\begin{bmatrix} R1 \\ \vdots \\ a_i Ri \\ \vdots \\ Rn \end{bmatrix} = \det\begin{bmatrix} R1 \\ \vdots \\ Ri \\ \vdots \\ Rn \end{bmatrix} + \det\begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ \sum_{j=1}^{n} a_j R_j \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix}$$

The last equality is by determinant rule (2) once again.

If M is not invertible, then M can be row reduced to a matrix with a row of zeros. In Problems 10 and 11, we show that the formula holds in this case as well. ■

EXAMPLE 48 In the matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, row two is the sum $[1\ 0\ 0] + [1\ 1\ 1]$. Use this fact and the formula of Theorem 23 to compute $\det M$.

Solution We may let $R1 = [1\,1\,1]$, $R2 = [1\,0\,0]$, $R2' = [1\,1\,1]$, and $R3 = [1\,0\,3]$. Theorem 23 tells us that

$$\det\begin{bmatrix} R1 \\ R2 + R2' \\ R3 \end{bmatrix} = \det\begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix} + \det\begin{bmatrix} R1 \\ R2' \\ R3 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} + \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The second of these matrices row reduces to a matrix with a row of zeros, so $\det(M)$ is the determinant of the first matrix. Applying row reduction, we get

$$\det M = \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} \xrightarrow[-R2+R1]{-R2+R3} \det\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow{R1 \leftrightarrow R2} \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Since we now have an upper triangular matrix, we see that $\det M = 3$. ■

Theorem 23 is often restated as follows: "The determinant of a matrix is an **additive function** of row i." This additive property together with determinant axiom (1) are often restated: "The determinant is a **linear function** of row i."

Using the Additive Property

EXAMPLE 49 Use the fact that

$$[1\ 2\ 3] = [1\ 0\ 0] + [0\ 2\ 0] + [0\ 0\ 3]$$

to help evaluate the determinant of the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix}$$

Solution We may write

$$\begin{aligned} \det M &= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \end{aligned}$$

By interchanging columns, we may convert our matrices to triangular form:

$$\begin{aligned} \det M &= -\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} - \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} - \det \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 4 & 3 \end{bmatrix} \\ &= -4 \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 18 \\ &= -4 + 6 - 18 = -16 \end{aligned}$$

■

C Row and Column Expansions

Computations similar to those of Example 51 give us a formula for the determinant of a 3-by-3 matrix. Theorem 23 justifies the first step that follows; the other steps are explained after the computation.

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$
\begin{aligned}
&= \det\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} - \det\begin{bmatrix} a_{12} & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{bmatrix} + \det\begin{bmatrix} a_{13} & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{bmatrix} \\
&= a_{11}\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{bmatrix} \\
&= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
\end{aligned}
$$

The second equal sign is the result of row reduction. For example, look at the two matrices whose first row is $[a_{11}\ 0\ 0]$. If a_{11} is not zero, subtracting the right multiples of row 1 from rows 2 and 3 gives the matrix with zeros below a_{11}. If, on the other hand, a_{11} is zero, then both determinants are determinants with rows of zeros, so they both equal zero. Notice that the minus sign on the middle determinant comes from interchanging rows. The third equal sign follows from determinant rule (1). The reason for the very last step is this. The function of the 2-by-2 matrix of a_{ij}'s given by

$$f\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{bmatrix}$$

satisfies all the axioms for a determinant function on 2-by-2 matrices. Therefore f must be the determinant function on 2-by-2 matrices. We have proved, in the 3-by-3 case, a formula called the formula for **expansion on the first row** for the computation of a determinant.

Computing a Determinant by Expansion on a First Row

From the row expansion formula above, it is just a small step to obtaining the **general row expansion formula**, which is stated as Theorem 21.

Theorem 21 If we use $A(i, j)$ to stand for the matrix obtained from the matrix A by deleting row i and column j, then for any choice of i,

$$\det A = \sum_{j=1}^{n} (-1)^i + j\ a_{ij} \det A(i, j).$$

Proof Essentially the computation given above in the 3-by-3 case for $i = 1$.
■

The matrix $A(i, j)$ is called the i, j **minor** of A; the term $(-1)^{i+j} \det(A(i, j))$ is called the **cofactor** of a_{ij}.

EXAMPLE 50 Compute the determinant

$$\det M = \det \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution We choose to expand on row 2 (for reasons that will become clear soon), so we apply our formula with $i = 2$. Thus we get

$$\begin{aligned} \det M &= (-1)^2 + 1 \ \cdot 1 \det \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} + (-1)^{2+2} \cdot 0 \det \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \\ & \quad + (-1)^{2+3} \cdot 2 \det \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \\ &= -1(3+2) + 0 - 1 \cdot 2 \cdot (4 - 9) \\ &= -5 + 10 = 5 \end{aligned}$$

Why did we expand on row 2? Because it had a zero, allowing us to finish the computation by using two 2 by 2 determinants rather than three. ■

Recall that what works for rows of a matrix works for columns as well. Thus we may expand on a column as well.

EXAMPLE 51 Compute the determinant of the matrix M below by expansion on column 3.

$$M = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 4 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

Solution

$$\det M = (-1)^{1+3} \cdot 3 \cdot \det \begin{bmatrix} 0 & 4 & 6 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} + 0 + 0 + (-1)^{4+3} \cdot 2 \cdot \det \begin{bmatrix} 1 & 3 & 1 \\ 0 & 4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} &= 3 \cdot (-1)^{1+3}(8-6) - 2 \cdot (-1)^{1+1}(8-6) \\ &= 3 \cdot 2 - 2 \cdot 2 = 2 \end{aligned}$$

■

Expansion of a determinant along a row or column that contains as many zeros as possible is a good technique for evaluating determinants of matrices with a fair number of zeros. Furthermore, we have shown that if there is a determinant function on n-by-n matrices, then it may be computed recursively in terms of determinants of $(n-1)$-by-$(n-1)$ matrices. This suggests that we should be able to use the formula for 2-by-2 determinants to prove that 3-by-3 determinants exist, and so on, getting an inductive proof that there is a determinant function. Theorem 22 does this.

Theorem 22 The formula for expansion of a determinant on column 1 satisfies rules (1)–(4) for a determinant function.

Proof Problem 10 of Section 10-5 tells us to show that a function satisfies determinant rules (1)–(4) if it satisfies rules (1), (4), and (2″), the rule that adding a row to an adjacent row does not change the determinant. We assume inductively that the expansion formula is a determinant function for $(n-1)$-by-$(n-1)$ matrices, and we note that in the case n = 2 it is the usual formula. Then, using the formula

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(A(i,1))$$

we see that $\det(I_n) = (-1)^{1+1} \cdot 1 \cdot \det I_{n-1}$, where I_n and I_{n-1} denote the n-by-n and $n-1$-by-$n-1$ identity matrices. Thus since $\det I_{n-1} = 1$ we get that $\det I_n = 1$. Multiplying row k of A by r multiplies a_{k1} but not $\det(A(k,1))$ by r; by the inductive hypothesis it also multiplies each $\det(A(i,1))$ with $i \neq k$ by r. Thus the formula satisfies rule (1). Finally, if we add row k to row $k+1$, we don't change any of the $a_{i1} \det(A,1)$ terms with $i \neq k$ and $i \neq k+1$ (because the $n-1$-by-$n-1$ determinants must satisfy rule (2)). If A' is the matrix we get from A by adding row k to row $k+1$, then

$$a'_{k1}\mathrm{det}A'(k,1) = a_{k1}\mathrm{det}A'(k,1) = a_{k1}(\mathrm{det}A(k,1) + \mathrm{det}A(k+1,1)) \qquad (1)$$

by the additivity property for determinants of $n-1$-by-$n-1$ matrices. Also

$$a'_{k+1,1} \det A'(k+1,1) = (a_{k+1,1} + a_{k1}) \det A(k+1,1) \qquad (2)$$

Now $a_{k1} \det A(k+1,1)$ in (1) is multiplied by $(-1)^k$, and the $a_{k1} \det A(k+1,1)$ in (2) is multiplied by $(-1)^{k+1}$ in the column expansion for A'; therefore, they cancel out and the column expansion for A' is identical with that for A. Thus the formula for expansion on the first column satisfies the determinant rules. ■

The value of Theorem 25 is this. Until we proved it, we did not know there *was* a determinant function, we simply knew how to compute it if there were one. Now we know that there is such a function, and since the rules determine its value, there is only one such function.

Concepts Review

1. The determinant of a product of elementary matrices is the ____________ of their determinants.
2. The determinant of M times N is ____________ .
3. We form the ____________ of a matrix by interchanging the rows and columns.
4. The ____________ property tells us that if three matrices are identical except in one row which in one matrix is the sum of the corresponding rows in the other two then the determinant of the first matrix is the ____________ of the determinants of the other two.
5. The determinant of the transpose of M is ____________ to the ____________ of M.

6. The transpose of the product is the ____________ of the transposes in ____________ ____________ .
7. A matrix is invertible if its determinant is ____________ .
8. When computing a determinant by row or column expansion it is useful to pick a row or column with as many ____________ as possible.
9. The formula for the ____________ of a determinant on ____________ ________ is

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A(i,j)$$

A. Exercises

For Exercises 1–18, let M, N, P, Q, R, and S be the matrices given below.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad R = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix} \quad S = \begin{bmatrix} 2 & 0 & -3 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. Find the determinants of M, N, and P.
2. Find the determinants of Q, R, and S.
3. Find the determinant of MN.
4. Find the determinant of QR.
5. Find the determinant of NP.
6. Find the determinant of QS.
7. Find the determinant of MP.
8. Find the determinant of SR.
9. Find the determinant of N^2.
10. Find the determinant of Q^2.
11. Find the determinant of P^2.
12. Find the determinant of S^2.
13. Find MN and compute its determinant to check Exercise 3.
14. Find QR and compute its determinant to check Exercise 4.
15. Find N^2 and compute its determinant to check Exercise 9.
16. Find Q^2 and compute its determinant to check Exercise 10.
17. Find the transpose of the matrices M, N, and P in Exercise 1. What is the determinant of each of these transposes?
18. Find the transpose of the matrices Q, R, and S in Exercise 2 and the determinant of each of these transposed matrices.

B. Exercises

19. Apply the method of Example 51 using row 1 in the way [1 2 3] was used for part (a), row 2 for part (b), and row 3 for part (c) to compute each of the determinants below.

(a) $\det \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$ (b) $\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ (c) $\det \begin{bmatrix} 0.5 & 6.1 & 2.2 \\ 2 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

20. Apply the method of Example 51 using the row containing a zero in the way the row [1 2 3] was used to evaluate each of the following determinants.

(a) $\det\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$ (b) $\det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & \frac{1}{2} \\ 2 & 2 & 2 \end{bmatrix}$ (c) $\det\begin{bmatrix} 2 & 6 & 4 \\ -2 & 2 & -2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

21. In each matrix below, use the fact that $[a, b, 0] = [a, 0, 0] + [0, b, 0]$ to help evaluate the determinant of the matrix in terms of a and b. Assume that neither a nor b is 0.

(a) $\begin{bmatrix} 1 & b & 0 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ a & b & 0 \\ 3 & 2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ a & b & 0 \end{bmatrix}$

22. In each matrix below, use the fact that $[c, 0, d] = [c, 0, 0] + [0, 0, d]$ to help evaluate the determinant of the matrix in terms of c and d. Assume that neither c nor d is zero.

(a) $\begin{bmatrix} c & 0 & d \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ c & 0 & d \\ 0 & 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & -1 \\ c & 0 & d \end{bmatrix}$

23. Devise a method similar to the method of Example 51, but using columns and use it to evaluate each determinant in Exercise 19 by using a column containing a zero.

24. Devise a method similar to the method of Example 51, but using columns and use it to evaluate each determinant in Exercise 20 by using a column containing a zero.

C. Exercises

25. Use row expansion on row 1 for part (a), row 2 for part (b), and row 3 for part (c) to compute each of the determinants below.

(a) $\det\begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$ (b) $\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ (c) $\det\begin{bmatrix} 0.5 & 6.1 & 2.2 \\ 2 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

26. Use row expansion on the row containing a zero to evaluate each of the following determinants.

(a) $\det\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$ (b) $\det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & \frac{1}{2} \\ 2 & 2 & 2 \end{bmatrix}$ (c) $\det\begin{bmatrix} 2 & 6 & 4 \\ -2 & 2 & -2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

27. Use column expansion on a column containing a zero to evaluate each of the determinants in Exercise 25.

28. Use column expansion on a column containing a zero to evaluate each of the determinants in Exercise 26.

In Exercises 29–36, evaluate the determinants given by any combination of methods you feel is appropriate.

29. $\det \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 3 \\ -1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 4 \end{bmatrix}$

30. $\det \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ 6 & 12 & -6 & 0 \\ 2 & 1 & 1 & 4 \\ 5 & 1 & 3 & 1 \end{bmatrix}$

31. $\det \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 2 & -1 \end{bmatrix}$

32. $\det \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 3 & 3 & 2 \\ 0 & 0 & 5 & 4 \\ 1 & 1 & 0 & -1 \end{bmatrix}$

33. $\det \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 \\ 1 & 3 & 0 & 1 \end{bmatrix}$

34. $\det \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 8 & -1 & -8 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

35. $\det \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}$

36. $\det \begin{bmatrix} 2 & 6 & 0 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

37. Derive a formula for the determinant of the upper triangular matrix

$$M = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

by expanding on the first column.

38. Derive a formula for the determinant of the lower triangular matrix

$$M = \begin{bmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{bmatrix}$$

by expanding on the last column.

39. Consider the matrix

$$M = \begin{bmatrix} 3 & 4 & 5 \\ 2r & 3r & 4r \\ -1 & 2 & 3 \end{bmatrix}$$

Expand on the second row to show that the determinant of M is r times the determinant of

$$N = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ -1 & 2 & 3 \end{bmatrix}$$

40. Consider the matrix

$$M = \begin{bmatrix} 3 & 4r & 5 \\ 2 & 3r & 4 \\ -1 & 2r & 3 \end{bmatrix}$$

Expand on the second column to show that the determinant of M is r times the determinant of the matrix N of Exercise 39.

Problems

1. Show that a function which satisfies rule (3) for determinants has the property that the function must be zero if the matrix has two equal rows.

2. Show how rules (1) and (3) and the additivity property of determinants allow you to derive rule (2).

3. Prove by induction that $\det(M_1 M_2 \cdots M_k) = \det(M_1)\det(M_2)\cdots\det(M_k)$.

4. Define the elementary column operations and describe the effect of each kind of elementary column operation on the determinant.

5. Show that determinant rule (1) and the additivity property together are equivalent to the *linearity* property:

$$\det \begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ rRi + sR'i \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix} = r\det \begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ Ri \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix} + s\det \begin{bmatrix} \text{---}R1\text{---} \\ \vdots \\ R'i \\ \vdots \\ \text{---}Rn\text{---} \end{bmatrix}$$

6. Use Theorem 21 to show that M is invertible if and only if M^t is invertible.

7. Show that if a sequence of elementary row operations reduces M to a matrix with a row of zeros, then the same sequence of elementary row operations reduces MN to a matrix with a row of zeros.

8. If the matrix M is invertible, then how does $M^{-1}(MN)$ relate to N? What do you know about a product of invertible matrices? What can you conclude about N if you know that M and MN are invertible?

9. On the basis of Problem 8, show that $\det MN = 0$ if either M or N is not invertible.

10. Suppose that it is possible to row reduce the matrix M to a matrix with a row of zeros without ever using row i in the reduction. Explain why all three determinants in the statement of Theorem 11 are zero.

11. If there is some way to reduce the matrix M to a matrix with a row of zeros, then there are numbers $a_1, a_2, \ldots, a_n$ such that $a_1R_1 + a_2R_2 + \cdots + a_nR_n = 0$.

    (a) If $a_i \neq 0$, explain why we may reduce a to a matrix with row i equal to zero without adding row i to another row or exchanging row i with another row.

    (b) If $a_i \neq 0$, explain why the left- and right-hand determinants in Theorem 11 are equal while the middle determinant is zero.

    (c) If $a_i = 0$, explain why Problem 10 completes the proof of Theorem 11.

Chapter 10
Review Exercises

For Exercises 1–12 below, use the matrices

$$R = [1\ 0\ 1\ 1] \quad S = [1\ -1\ -1\ 2] \quad T = [1\ -2]$$

$$A = \begin{bmatrix} 1 & 4 \\ -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 & 0 & -1 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Find $R + S$, $2 \cdot R$, $-3 \cdot S$.
2. Find $A + E$.
3. Find $D(A + E)$ and $DA + DE$ and explain why we should expect the two results to be the same.
4. Find IE and EI.
5. Find OE and $O + E$.
6. Find $(SB)C$ and $S(BC)$ and explain why we should expect the results to be the same.
7. Find D^2. Find D^4. What will the entry in row 1 and column 2 of D^n be for any positive integer n?
8. Which of the matrix products SC, AB, $(A + E)C$, $(AD)E$ and AS are defined?
9. Find DE and ED. What general principle is illustrated by this computation?
10. Write down the 4-by-4 identity matrix and the product IC. What will the product IB be?
11. What is B_{32}? What is B_{23}? What is C_{31}?
12. Find the value of the following.

$$\sum_{i=1}^{2} A_{2k} E_{k1}$$

In which row and column of AE do we find this entry?

13. Write the system of equations below as a matrix equation.

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & x_3 & = & -2 \\ 2x_1 & & & - & x_3 & = & 3 \\ 4x_1 & - & 5x_2 & + & 2x_3 & = & -3 \end{array}$$

14. Write down the matrix equation which corresponds to the system of equations consisting of the first 2 equations of Exercise 13.
15. Circle the pivotal entries in the matrices M and N that follow.

$$M = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 3 & 0 & 4 & 1 \\ 0 & 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 1 & 2 & 3 & 0 & 1 \end{bmatrix}$$

16. Which, if any, of the matrices in Exercise 15 are row reduced?

17. Row reduce the matrices M and N that follow

$$M = \begin{bmatrix} 2 & 0 & 4 & -2 \\ 1 & 3 & -1 & 2 \\ 0 & 1 & 2 & -2 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 0 & 4 & -2 \\ 1 & 3 & -1 & 2 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

18. Write the augmented matrices corresponding to the systems of equations below.

(a) $\begin{aligned} 2x_1 \quad\quad + 4x_3 - 2x_4 &= 2 \\ x_1 + 3x_2 - x_3 + 2x_4 &= 7 \\ x_2 + 2x_3 - 2x_4 &= 5 \end{aligned}$ b) $\begin{aligned} 2x_1 \quad\quad + 4x_3 - 2x_4 &= 2 \\ x_1 + 3x_2 - x_3 + 2x_4 &= 7 \\ - x_2 + x_3 - x_4 &= 1 \end{aligned}$

19. Row reduce the augmented matrices you wrote in Exercise 18.

20. Which variables are pivotal in the systems of equations corresponding to the two row reduced augmented matrices in Exercise 19?

21. Find all solutions to the two systems of equations in Exercise 18.

22. Solve the system of equations in Exercise 13 by row reducing an augmented matrix.

23. For each augmented matrix below, specify whether the system of equations corresponding to it has no solutions, one solution, or infinitely many solutions.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

In Exercises 24–27, find all solutions to the system of equations given.

24. $\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 + 2x_2 - x_3 &= 2 \\ -x_1 - 2x_2 + 3x_3 &= 4 \end{aligned}$ 25. $\begin{aligned} 2x_1 + 4x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + x_3 &= 2 \\ 2x_1 + 4x_2 \quad\quad &= 2 \end{aligned}$

26. $\begin{aligned} x_1 + x_2 - x_3 - 3x_4 &= -1 \\ -2x_1 - 4x_2 + x_3 + 3x_4 &= 0 \\ 2x_1 + 4x_2 - 2x_3 - 6x_4 &= -2 \end{aligned}$ 27. $\begin{aligned} 2x_1 + 2x_2 - x_3 + x_4 &= 3 \\ x_1 + x_2 + x_3 + x_4 &= 3 \\ -3x_1 - 3x_2 + x_3 - x_4 &= 3 \\ x_1 + x_2 - 2x_3 + 2x_4 &= 5 \end{aligned}$

28. Write down the following 3-by-3 elementary matrices

(a) $E(3R2)$ (b) $E(-R3+R2)$ (c) $E(3R3+R1)$ (d) $E(R1 \leftrightarrow R3)$

29. Show the effect of multiplying each of the elementary matrices of Exercise 28 on the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

30. Write down the inverses of each of the elementary matrices in Exercise 29.

31. Show that the matrix M below has an inverse by finding a sequence of row operations which row reduces it to the identity.

$$M = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$

32. Show that the matrix N below does not have an inverse.

$$N = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ 2 & 4 & 0 \end{bmatrix}$$

33. Write down a sequence of elementary matrices whose product is the inverse of the matrix M in Exercise 31.

34. Write down the 3-by-3 augmented matrix you could row reduce in order to compute the inverse of the matrix M in Exercise 31.

35. Use row reduction to compute the inverse of the matrix M in Exercise 31.

36. Use the inverse matrix you computed in Exercise 35 to solve the system of equations in Exercise 24.

37. Determine whether the following matrices are invertible and find their inverses if they are.

(a) $M = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ (b) $N = \begin{bmatrix} 2 & 4 & 2 \\ 3 & -2 & 1 \\ 5 & 2 & 3 \end{bmatrix}$ (c) $P = \begin{bmatrix} 2 & 4 & 2 \\ 3 & -2 & 1 \\ 5 & 2 & 4 \end{bmatrix}$

38. Find the determinants of the matrices D and E of Exercises 1–12.

39. Find the determinant of the matrix DE, using D and E from Exercise 38.

40. Find the determinant of the matrix B of Exercises 1–12.

41. Find the determinant of the matrix M of Exercise 29.

42. Find the determinant of the matrix P of Exercise 37.

43. A matrix is *lower triangular* if all the entries above its main diagonal (that is, all a_{ij} with $i < j$) are zero. Explain why the determinant of a lower triangular matrix with ones in the main diagonal is one.

44. Using the matrix B of Exercises 1–12, and the matrix M of Exercise 37, find $\det(MB)$.

45. Using the matrices M and B of Exercise 44, find $\det(MB^t)$ and $\det(M^tB)$.

46. Write down the transpose of the matrix P of Exercise 37.

47. Write down the 3-by-3 elementary column matrices specified.
(a) $E(3C_1)$ (b) $E(2C1 + C2)$ (c) $E(C2 \leftrightarrow C3)$

48. Use any method you feel is appropriate to determine the determinant of the matrix M below. Is M invertible?

$$\begin{bmatrix} 0 & 1 & 4 & 3 & 0 \\ 2 & -1 & 3 & 5 & 8 \\ 0 & 3 & 2 & -3 & 3 \\ 0 & 4 & -2 & 0 & 0 \\ 0 & -5 & 1 & -2 & 1 \end{bmatrix}$$

49. Prove by induction that $\det(M_1 \cdot M_2 \cdot \ldots \cdot M_k) = \det(M_1) \cdot \ldots \cdot \det(M_k)$.

In Exercises 50 and 51, draw the graph or multigraph whose adjacency matrix is given.

50. $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ 51. $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 1 & 0 \\ 1 & 3 & 0 & 4 & 0 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$

In Exercises 52 and 53, draw the digraph or multidigraph whose adjacency matrix is given.

52. $$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

53. $$\begin{bmatrix} 0 & 3 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

55. Using matrix arithmetic, find the number of 4-edge walks form vertex 1 to vertex 2 and from vertex 1 to vertex 5 in the graph of Exercise 50. Explain from the drawing why one of these answers is 0.

56. Use matrix arithmetic to determine which pairs of vertices in the digraph of Exercise 52 have walks of length 3 between them.

57. Use Warshall's algorithm to find the transitive closure of the relation of the graph in Exercise 50. What does this tell you about the connected components of the graph?

58. Use Warshall's algorithm to find the transitive closure of the digraph of Exercise 52. Use this to tell what vertices are reachable from each vertex of the digraph.

59. Regard the entries of the matrix of Exercise 51 as weights on edges of a graph and use Floyd's algorithm to find the distance from each vertex to each vertex which can be reached form it.