Power series and Taylor polynomials

Kenneth P. Bogart

April 5, 2001

1 Polynomial solutions of differential equations

Suppose we have a differential equation that we suspect may have a polynomial solution. It is often pretty easy to follow up on such a suspicion. For example, it is natural to guess that the differential equation

$$y'' = -x \tag{1}$$

has a polynomial solution, and in fact you may have an idea of how you would find one. Here we will illustrate a general method. If y is a polynomial function of x, then

$$y = a_0 + a_1x + a_2x^2 + \cdots a_nx^n, \tag{2}$$

which we also write as

$$y = \sum_{i=0}^{n} a_ix^i.$$

Then

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1},$$

and

$$y'' = 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots + n(n-1)a_nx^{n-2}. \tag{3}$$

If we also know that $y'' = -x$, then we know that

$$\begin{aligned} y'' &= 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + \cdots + n(n-1)a_nx^{n-2}. \\ &= \qquad\qquad -x. \end{aligned}$$

Now we know that two polynomials are equal if and only if for each i the coefficient of x^i is the same. This tells us that

$$\begin{aligned} 2a_2 &= 0 \\ 3 \cdot 2a_3 &= -1 \\ 4 \cdot 3a_4 &= 0 \\ &\vdots \\ n(n-1)a_n &= 0 \end{aligned}$$

From this we may conclude that if the y in Equation 2 is a solution to the differential equation 1, then

$$a_2 = a_4 = a_5 = \cdots = a_n = 0$$

and (with a little arithmetic) $a_3 = \frac{-1}{6}$, while a_0 and a_1 may be anything we like. Thus every possible polynomial solution has the form

$$y = a_0 + a_1 x - \frac{1}{6} x^3.$$

Clearly if we take the second derivative of any such y, we will get $-x$. Thus we have found the most general polynomial solution that our differential equation can have; in fact, though we won't show it until later, this is the most general solution our differential equation has.

2 Power series solutions

Since this experiment was so successful, it is natural to ask what would happen if we try to find a polynomial solution to the differential equation

$$y'' = -y \tag{4}$$

by the same method. Again we suppose y has the form in Equation 2, and so its second derivative is given by Equation 3. Now the equation $y'' = -y$ becomes

$$\begin{aligned} y'' &= (2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \cdots + n(n-1)a_n x^{n-2}). \\ &= -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - \cdots - a_n x^n. \end{aligned}$$

Again using the fact that two polynomials are equal if and only if each power of x has exactly the same coefficient in each, we may write

$$\begin{aligned} 2a_2 &= -a_0 \\ 3 \cdot 2a_3 &= -a_1 \\ 4 \cdot 3a_4 &= -a_2 \\ 5 \cdot 4a_5 &= -a_3 \\ &\vdots \\ n(n-1)a_n &= -a_{n-2}. \end{aligned}$$

From this we see that

$$a_2 = -\frac{a_0}{2} \tag{5}$$

$$a_3 = -\frac{a_1}{3 \cdot 2} \tag{6}$$

$$a_4 = -\frac{a_2}{4 \cdot 3} \tag{7}$$

$$a_5 = -\frac{a_3}{5 \cdot 4} \tag{8}$$

$$\vdots$$

$$a_n = -\frac{a_{n-2}}{n(n-1)}. \tag{9}$$

By substituting Equations 5 and 6 into Equations 7 and 8, we get

$$a_4 = \frac{a_0}{4 \cdot 3 \cdot 2} = \frac{a_0}{4!} \tag{10}$$

$$a_5 = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 3} = \frac{a_1}{5!}. \tag{11}$$

Equations 10 and 11 and their natural extensions tell us that

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 - \cdots. \tag{12}$$

Equation 12 may be rewritten as

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right). \tag{13}$$

Now you may be asking what happened to the last term of the polynomial, the x^n term in Equations 12 and 13. Was it just the case that we left the nth term out because we didn't know whether n was even or odd? No, something more subtle is happening. We can see the problem easily by considering the special case with $n = 8$, in which case Equation 13 becomes

$$y = a_0(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + +\frac{x^8}{8!}) + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} + -\frac{x^7}{7!}). \tag{14}$$

When we take the second derivative of the y in Equation 14 we get the polynomial

$$y = a_0(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}) + a_1(-x + \frac{x^3}{3!} - \frac{x^5}{5!}). \tag{15}$$

At first Equation 15 looks like the negative of Equation 14, but in fact, it does not have the two highest power terms of the negative of Equation 14. Thus we certainly do not have a polynomial solution with $n = 8$. We picked $n = 8$ to have a clear example, but in a similar way, no matter how we choose n, we will have failed to find a polynomial solution. However, all is not lost. We can see that if we chose a larger value of n our problem would occcur with larger powers of x. It makes us think that if we could only pick $n = \infty$, all our problems would go away. This is, in effect, what we will do.

Suppose we interpret the sums

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

from Equation 13 as infinite sums, sums that go on forever. Then so long as the two infinite sums make sense and so long as the derivative of an infinite sum of this type is the sum of its derivatives, we get from Equation 13

$$y'' = a_0(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots) + a_1(-x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots). \tag{16}$$

Clearly $y'' = -y$, just as desired.

Now we have a big question left. Do the two infinite sums in Equation 13 make sense. Does it mean anything to write down an infinite sum? The answer is, it all depends, but there is no reason we can't write such a sum down and then figure out whether or not it is nonsense.

For example, in an algebra course you learn (well, you are exposed to it anyhow) the formula for the sum of a finite geometric series:

$$1 + r + r^2 + \cdots r^n = \frac{1 - r^{n+1}}{1 - r}$$

(when $r \neq 1$) and the formula for the sum of an infinite geometric series

$$1 + r + r^2 + \cdots r^n + \cdots = \frac{1}{1 - r}$$

(when $-1 < r < 1$).

Thus, for example,

$$1 + \frac{1}{3} + (\frac{1}{3})^2 + \cdots (\frac{1}{3})^5 = \frac{1 - (\frac{1}{3})^6}{1 - \frac{1}{3}} = \frac{3}{2}(1 - (\frac{1}{3})^6 = 1092/729 = 364/243.$$

This is about 1.48. More easily,

$$1 + \frac{1}{3} + (\frac{1}{3})^2 + \cdots (\frac{1}{3})^n + \cdots = \frac{1}{1 - (\frac{1}{3})} = \frac{3}{2}.$$

Thus we have one infinite sum that makes sense. In fact, the infinite geometric series above makes sense only if r is between -1 and 1. Clearly it doesn't make sense if $r = 1$. So we see it is one thing to write down a series, and another thing to make sense of it.

An indicated sum of the form

$$A_0 + A_1 + A_2 + \cdots A_n + \cdots,$$

which we also write as

$$\sum_{i=o}^{\infty} A_i$$

is called an *infinite series*. An infinite series is really just a bunch of symbols we write down. Sometimes it makes sense and sometimes it doesn't. How do we decide which is the case? The geometric series shows how we can decide. If $|r| < 1$, then the as n gets larger and larger the term r^n gets closer and closer to zero. In fact it approaches 0 as a limit, so

$$\frac{1 - r^{n+1}}{1 - r}$$

approaches

$$\frac{1}{1-r}$$

as a limit.

Thus the sequence

$$\begin{array}{l} 1 \\ 1+r \\ 1+r+r^2 \\ 1+r+r^2+r^3 \\ \vdots \\ 1+r+r^2+\cdots+r^n \\ \vdots \end{array}$$

approaches the limit $\frac{1}{1-r}$ as n becomes infinite, so long as $|r| < 1$. If r=1, the sequence

$$\begin{array}{l} 1 \\ 1+1 \\ 1+1+1 \\ \vdots \\ n+1 \\ \vdots \end{array}$$

becomes infinite, as does the corresponding sequence if $r > 1$. Thus the finite sums do not approach a limit and the infinite sum makes no sense as a number. Can you see why the sequence does not approach a limit when $r = -1$ or $r < -1$?

By analogy, we say the infinite series

$$A_0 + A_1 + A_2 + \cdots A_n + \cdots$$

converges to the *value* A if the sequence

$$\begin{array}{l} A_0 \\ A_0 + A_1 \\ A_0 + A_1 + A_2 \\ \vdots \\ A_0 + A_1 + A_2 + \cdots + A_n \\ \vdots \end{array}$$

has the limit A as n becomes infinite. At present, the only examples we have of convergent series are geometric series for various values of r between -1 and 1. However, we will shortly have a number of very interesting convergent series.

3 Power Series

An infinite series of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_i x^i + \cdots = \sum_{i=1}^{\infty} a_i x^i \tag{17}$$

or

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_i(x-a)^i + \cdots = \sum_{i=1}^{\infty} a_i (x-a)^i \tag{18}$$

(where a represents a fixed real number) is called a *power series*. The infinite geometric series actually is an example of a power series; just think of what we get if we replace r by x:

$$1 + x + x^2 + \cdots + x^i + \cdots = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

(when $-1 < x < 1$). Thus we have one example of a power series that represents a function of x. This leads us to ask which functions that we know about do or don't have power series representations. Also, if a function does have a power series representation, how do we find it. Surprisingly, this second question is the easier one to answer.

We have given two different forms for a power series in Equations 17 and 18. Equation 18 includes a constant a. Notice that if we take a equal to 0, then we get the the kind of power series shown in Equation 17. Thus for our discussion of how to find a power series representation if there is one, we will just deal with the form we gave for a power series in Equation 18, the one including a.

Suppose f has the power series representation

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_i(x-a)^i + \cdots = \sum_{i=1}^{\infty} a_i (x-a)^i.$$

Then what is $f(a)$? Since $(a-a)$, $(a-a)^2$, $(a-a)^3$, etc., are all zero, $f(a) = a_0$. That tells us how to find a_0. How can we find a_1? Recall when we were taking derivatives of general polynomials above, taking the first derivative got rid of the a_0 term and left the a_1 term without a variable. That is, assuming our function has a derivative at $x = a$, and assuming we can find it by taking the derivatives of the terms in our infinite sum and summing them, we get

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots.$$

Then $f'(a) = a_1$, giving us a way to find a_1. To find a_2, then, we take the second derivative to get

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 + 5 \cdot 4a_5(x-a)^3 + \cdots.$$

This gives us $f''(a) = 2a_2$, or

$$a_2 = \frac{f''(a)}{2}.$$

Taking third derivatives gives us

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-a) + 5 \cdot 4 \cdot 3a_5(x-a)^2 + 6 \cdot 5 \cdot 4a_6(x-a)^3 + \cdots.$$

This gives us $f'''(a) = 3!a_3$, which we solve for a_3 to get

$$a_3 = \frac{f'''(a)}{3!}.$$

Continuing in this way, and using $f^{(i)}$ to stand for the ith derivative of f, we get in general

$$a_i = \frac{f^{(i)}(a)}{i!}.$$

This gives us the following remarkable theorem (which applies whenever we have a function represented by a power series we may differentiate term by term).

Theorem 1 *If f has infinitley many derivatives and is equal to a power series, then*

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(i)}(a)}{i!}(x-a)^i + \cdots \\ &= \sum_{i=0}^{\infty} \frac{f^{(i)}}{i!}(x-a)^i. \end{aligned} \tag{19}$$

4 Taylor series and Taylor polynomials

A series of the form in Equation 19 is called a *Taylor series* of the function f around $x = a$. If we take the first $n+1$ terms (including the a_0 term) of a Taylor series for a function f, then we get the *Taylor Polynomial*

$$\begin{aligned} P_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{i!}(x-a)^n \\ &= \sum_{i=0}^{n} \frac{f^{(i)}}{i!}(x-a)^i. \end{aligned} \tag{20}$$

Notice that a Taylor polynomial is always a function of x, whether or not the corresponding Taylor series represents one. Also notice that when $n = 1$ the Taylor polynomial is just the right hand sid of the equation for a tangent line to the graph of f and $x = a$. Just as we used the limit of finite geometric series to make sense of the infinite geometric series, we make sense of the infinite Taylor series of a function f by thinking of it as the limit of the Taylor polynomials for f.

Example 1 *Compute the Taylor polynomial P_4 for $\sin x$ around $x = 0$.*

We first note that if $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \cos x \text{ so } f'(0) = 1 \\ f''(x) &= -\sin x \text{ so } f''(0) = 0 \\ f'''(x) &= -\cos x \text{ so } f'''(0) = -1 \\ f''''(x) &= \cos x \text{ so } f''''(0) = 0 \end{aligned}$$

This gives us

$$P_4(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}.$$

Example 2 *Compute the Taylor polynomial P_3 of $\sqrt{x-1}$ around $x = 5$.*

We know that

$$\begin{aligned} f(x) &= (x-1)^{\frac{1}{2}} \text{ so that } f(5) = 2 \\ f'(x) &= \frac{1}{2}(x-1)^{-\frac{1}{2}} \text{ so that } f'(5) = \frac{1}{4} \\ f''(x) &= -\frac{1}{4}(x-1)^{-\frac{3}{2}} \text{ so that } f''(5) = -\frac{1}{32} \\ f'''(x) &= \frac{3}{8}(x-1)^{-\frac{5}{2}} \text{ so that } f'''(5) = \frac{3}{256} \end{aligned}$$

Therefore $P_3(x) = 2 + \frac{(x-5)}{4} - \frac{(x-5)^2}{64} + \frac{(x-5)^3}{512}$.

5 Exercises

1. Use the method of assuming you have a polynomial solution to the differential equation $y''' = x$ and substituting the general form of the polynomial into the equation to find the most general polynomial solution.

2. Use the method of assuming you have a polynomial solution to the differential equation $y' = y$ and discovering that the solution must be a power series instead to find the most general power series solution. Check by taking derivatives that your power series does satisfy the diffferential equation. You actually already know the general solution to this differential equation. Make a conjecture about what function this power series represents. What is the Taylor series around $x = 0$ of the function that you conjectured?

3. Use the method of assuming you have a polynomial solution to the differential equation $y' = -y$ and discovering that the solution must be a power series instead to find the most general power series solution. Check by taking derivatives that your power series does satisfy the differential equation. You actually already know the general solution to this differential equation. Make a conjecture about what function this power series represents. What is the Taylor series around $x = 0$ of the function that you conjectured?