Radius and Open Interval of Convergence

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1 Convergence of $\sin x$.

We have discussed the remainder formula for Taylor polynomials in class, and it is worked over in Calculus, by Adams, in some detail. As one last example, our formula in class for the Taylor polynomial P_{2n+1} of $\sin x$ around 0 can be expressed algebraically as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \pm \frac{\sin X}{(2n+2)!} x^{2n+2},$$

where X is some constant between 0 and x and the plus or minus sign depends on where we are in the rotation of plusses and minuses in the derivatives of $\sin x$. (In fact, we get a plus sign whenever $2n+2$ is a multiple of 4, but whether or not we have a plus sign is unimportant.) What is important is that no matter what X is, $\sin X \leq 1$. That means that the last term, the remainder is no more than $x^{2n+2}/(2n++2)!$. It is not too hard to see that the limit of this remainder is zero, no matter how big x is. Note that x^{2n+2} is the product of $n+1$ terms each equal to x^2. For any n larger than or equal to x^2, the terms $n+1$, $n+2$, $\ldots$, $2n+2$ are all larger than x^2, and there are $n+1$ such terms. Dividing one of them into each of the x^2s in x^{2n+2} gives us a number less than 1, so this remainder is less than $1/n!$. When n is larger, the remainder is still less than $n!$ (a lot less). Thus the remainder approaches zero. The reason why this argument makes sense is that the value of x is independent of n, so as n gets larger and larger, x stays the same. Therefore for any x, the power series for $\sin x$ converges to $\sin x$.

2 Radius of Convergence

The idea of a radius of convergence helps us understand the difference between a power series like the geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \tag{1}$$

which converges for all x in the open interval

$$(-1, 1) = \{x1 < x < 1\},$$

and a power series like

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \cdots = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!}, \tag{2}$$

which we just saw converges for all real numbers x. In other words, it converges for all x in the open interval

$$(-\infty, \infty) = \{x | x \text{ is a real number}\}.$$

The two intervals we have mentioned are called the *intervals of convergence* of the power series. We also say the *radius of convergence* of Equation 1 is 1 and the *radius of convergence* of Equation 2 is ∞.

In the book by Calculus (p554, Theorem 17), Adams shows why for each power series, it converges at only one point, or it converges for all points inside an interval $(a-R, a+R)$ and diverges for all points outside the corresponding closed interval $[a-R, a+R]$. The number R is called the radius of convergence of the power series. There is a theorem that gives us a way to find out the radius of convergence of many different power series. When it applies, this theorem also substitutes for Theorem 17 in Adams book. We state it and show how to use it before developing the tools we need to prove it.

Theorem 1 Radius of Convergence Theorem. *If* $\lim_{n\to\infty} |a_{n+1}/a_n|$ *exists and equals the nonnegative number* L*, then the power series* $\sum_{i=0}^{\infty} a_i(x-a)^i$ *converges for all* x *in the open interval* $(a - \frac{1}{L}, a + \frac{1}{L})$ *and diverges for all* x *outside the closed interval* $[a - \frac{1}{L}, a + \frac{1}{L}]$.

Notice that the theorem says nothing about what happens at the endpoints of the interval. We call the open interval in the Theorem the *open interval of convergence* of the power series. It turns out it is possible to cook up examples that do any combination of converging or diverging we would like at these endpoints. In this course, we are simply going to ignore the behavior at endpoints of the interval of convergence. Note also that if $L = 0$, then we consider $\frac{1}{L}$ to be ∞, and the interval of convergence is $(-\infty, \infty)$. We will use the letter R to stand for $\frac{1}{L}$ and call R the *radius of convergence* of the power series.

When we apply the theorem to the geometric series, we find because each a_i is one, then L is 1. Therefore $R = 1$ as well. Notice that the theorem *does not* tell us that the geometric series converges to $\frac{1}{1-x}$. Surprisingly, it is possible to cook up functions whose Taylor series converge to something else, but we won't encounter such functions in this course.

The Radius of Convergence Theorem as it is stated actually does not apply to the Taylor series for $\sin x$ around zero, because the powers of x go up by two each time instead of by one as the theorem requires. Alternately, you might say it does not apply to the series for $\sin x$ around zero because every other a_i is zero, so the limit does not exist.

Example 1 What is the radius and open interval of convergence of the power series

$$\sum_{i=0} i2^i(x-1)^i?$$

$$\lim_{i\to\infty} |\frac{(i+1)2^{i+1}}{i2^i}| = \lim_{i\to\infty} |(1+\frac{1}{i})\cdot 2| = 2 = L$$

Therefore $R = \frac{1}{2}$ and the open interval of convergence is $(\frac{1}{2}, \frac{3}{2})$.

More examples of finding the radius of convergence of a power series by this method may be found on pages 555 and 556 of the book Calculus, by Robert Adams. We recommend skipping over discussions of how to determine whether or not a series converges at the endpoints of the interval of convergence unless you have the time to go through the earlier parts of Chapter 9.

We plan to complete this section of notes with a discussion of a proof of Theorem 1 that does not rely on much of the early discussion in Chapter 9. With luck the more complete set of notes will be available by sometime on April 8.