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1. (24) Solve the following initial value problems.

(a) $$\begin{cases} \dfrac{dy}{dx} + \sin(x)y^2 = 0 \\ y(0) = 1. \end{cases}$$

ANS: This equation is separable. Rewrite the equation as

$$\begin{aligned} \frac{dy}{dx} &= -\sin(x)y^2. \text{ Now separate variables to get} \\ -\frac{1}{y^2}\,dy &= \sin(x)\,dx. \text{ After integrating we get} \\ \frac{1}{y} &= -\cos(x) + C, \text{and} \\ y(x) &= \frac{1}{C - \cos(x)}. \end{aligned}$$

Since $y(0) = 1$, it follows that $C = 2$, and the final answer is

$$\boxed{y(x) = \frac{1}{2 - \cos(x)}.}$$

(b) $$\begin{cases} \dfrac{dy}{dx} + \sin(x)y^2 = 0 \\ y(0) = 0. \end{cases}$$

ANS: By separation of variables in the previous part, we got $y(x) = \dfrac{1}{C - \cos(x)}$. If $y(0) = 0$, then this gives us the equation $0 = 1$. However when we divided by y^2, to separate variables, we assumed $y \neq 0$. In fact, if $y = 0$, then $\dfrac{dy}{dx} = 0$ and $\sin(x)y^2 = 0$, so we have the constant solution $y = 0$.

(c) $$\begin{cases} \dfrac{dy}{dx} + \dfrac{2}{10 + x}y = 3 \\ y(0) = 1. \end{cases}$$

ANS: This equation is linear with $p(x) = 2/(10+x)$. Thus $\mu(x) = 2\log(10+x)$ and $e^{\mu(x)} = (10+x)^2$.

After multiplying both sides by $(10+x)^2$, we get

$$\begin{aligned}(10+x)^2y' + 2(10+x)y &= 3(10+x)^2, \text{ and therefore}\\ \big((10+x)^2y\big)' &= 3(10+x)^2, \text{ and after integrating}\\ (10+x)^2y &= (10+x)^3 + C, \text{ and hence}\\ y(x) &= 10 + x + \frac{C}{(10+x)^2}.\end{aligned}$$

Since $y(0) = 1$ implies $10 + \frac{C}{100} = 1$, we see that $C = -900$. The final answer is therefore

$$\boxed{y(x) = 10 + x - \frac{900}{(10+x)^2}.}$$

(d) $\begin{cases} \dfrac{dy}{dx} + \dfrac{2}{10+x}y = 3 \\ y(0) = 0. \end{cases}$

ANS: Here there is not a constant solution with $y = 0$. We saw by use of our integrating factor in the previous part that

$$y(x) = 10 + x + \frac{C}{(10+x)^2}.$$

If $y(0) = 0$, we get $-10 = \dfrac{C}{10^2}$, or $C = -1000$. Thus our final answer is

$$\boxed{y(x) = 10 + x - \frac{1000}{(10+x)^2}.}$$

2. (24) Consider the second order differential equation

$$(*) \qquad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0.$$

(a) Find the general solution to $(*)$.

ANS: The auxillary equation is $r^2 + 2r + 2 = 0$ with complex roots

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

By a Theorem stated in lecture, the general solution is

$$\boxed{y(x) = Ae^{-x}\cos(x) + Be^{-x}\sin(x).}$$

(b) Find the solution which satisfies the initial conditions $y(0) = 2$ and $y'(0) = 3$.

ANS: If $y(0) = 2$, we must have $A = 2$. Since

$$y'(x) = -Ae^{-x}\cos(x) - Ae^{-x}\sin(x) - Be^{-x}\sin(x) + Be^{-x}\cos(x),$$

$y'(0) = 3 = -A + B = -2 + B$. So $B = 5$, and the solution is

$$\boxed{y(x) = 2e^{-x}\cos(x) + 5e^{-x}\sin(x).}$$

(c) Find one solution to the second order differential equation

$$(\dagger) \qquad \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2x^2.$$

ANS: It makes sense to assume that a polynomial of degree no more than 2 is a solution. Suppose $y = a_0 + a_1x + a_2x^2$. Then

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2a_2 + 2a_1 + 4a_2x + 2a_0 + 2a_1x + 2a_2x^2 = 2x^2.$$

This gives us $2a_2 = 2$, so that $a_2 = 1$. It gives us $4 \cdot a_2 + 2a_1 = 0$, so that $a_1 = -2a_2 = -2$. Finally, it gives us that $2a_2 + 2a_1 + 2a_0 = 0$, so that $2a_0 = -2a_1 - 2a_2$, or $a_0 = -a_1 - a_2 = 1$. Thus one solution to the differential equation is $y_p = x^2 - 2x + 1$.

(d) Find the general solution to $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 2x^2$.

ANS: The general solution is $y(x) = y_p(x) + y_h(x)$ where y_h is the general solution from part (a) and y_p is the particular solution from part (c). Thus the general solution is

$$\boxed{y(x) = x^2 - 2x + 1 + Ae^{-x}\cos(x) + Be^{-x}\sin(x).}$$

3. (7) Find the MacLaurin series for

$$f(x) = x^2\cos(2x).$$

To get full credit you must provide at least four nonzero terms and a formula for the general term of the series.

ANS: Recall that for all x, $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$. Therefore, $\cos(2x) = 1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + \dots$. Thus

$$\boxed{x^2\cos(2x) = x^2 - \frac{2^2x^4}{2!} + \frac{2^4x^6}{4!} - \frac{2^6x^8}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n+2}}{(2n)!}.}$$

4. (16) Let $f(x) = \cos(x)$.

(a) Find the Taylor polynomial of degree 2 for f about $x = 0$.

ANS: Well, $f'(x) = -\sin(x)$ and $f''(x) = -\cos(x)$. Since $P_2(x) := f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$, we get

$$\boxed{P_2(x) = 1 - \frac{x^2}{2}.}$$

(b) Use your answer from part (a) to estimate $\cos(\frac{1}{2})$.

ANS: $\boxed{P_2\left(\frac{1}{2}\right) = 1 - \frac{1}{8} = \frac{7}{8}}$.

(c) Find the best upper bound you can for the absolute value of the error in your estimate in part (b). (That is, you should find the smallest number which you can show is larger than the absolute value of the difference between $\cos(\frac{1}{2})$ and your estimate in part (b).)

ANS: To get the best answer (and therefore the most credit), you should note that $P_2(x) = P_3(x)$. Thus

$$\cos\left(\frac{1}{2}\right) - P_2\left(\frac{1}{2}\right) = E_3\left(\frac{1}{2}\right) = \frac{f^{(4)}(c)}{4!}\left(\frac{1}{2} - 0\right)^4 = \frac{\cos(c)}{24}\frac{1}{16},$$

and $c \in (0, \frac{1}{2})$. Since $|\cos(c)| \leq 1$ for any value of c and since $24 \times 16 = 384$,

$$\boxed{\left|E_3\left(\frac{1}{2}\right)\right| \leq \frac{1}{384}.}$$

(*Note that to get full credit, you must explain how you made the estimate above; i.e.,* $|\cos(c)| \leq 1$.)

If you use E_2 instead of E_3, you still get an upper bound, but a larger, and therefore less desirable one:

$$E_2\left(\frac{1}{2}\right) = \frac{f'''(c)}{3!}\left(\frac{1}{2} - 0\right)^3 = \frac{\sin(c)}{6 \cdot 8}.$$

If you just observe the $|\sin(c)|$ is always less than 1 you get an upper bound of $\boxed{\frac{1}{48}}$. If you note that for $c \in (0, \frac{1}{2})$, $0 < \sin(c) < \sin(\frac{1}{2}) < \sin(\frac{\pi}{6}) = \frac{1}{2}$, then you get a *better* upper bound of $\boxed{\frac{1}{96}}$. I also accepted $\sin(\frac{1}{2})/48$.

(d) (This part of problem 4 is a bit subtle. You may want to come back to it after you've completed the rest of the exam.) Find the shortest interval which you can guarantee contains the exact value of $\cos(\frac{1}{2})$. (If you wish, you may use the facts that $\frac{1}{2} < \frac{\pi}{6}$, that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and that $\sin(\frac{\pi}{6}) = \frac{1}{2}$.)

ANS: To get the best possible answer, we recall from the previous part that $E_3(\frac{1}{2}) = \frac{\cos(c)}{384}$ with $c \in (0, \frac{1}{2})$. Since

$$\frac{\sqrt{3}}{2} = \cos\Big(\frac{\pi}{6}\Big) < \cos\Big(\frac{1}{2}\Big) < \cos(c) < \cos(0) = 1,$$

we have $\frac{\sqrt{3}}{768} \le E_3(\frac{1}{2}) \le \frac{1}{384}$. Since $\cos(\frac{1}{2}) = \frac{7}{8} + E_3(\frac{1}{2})$, we get

$$\boxed{\cos\Big(\frac{1}{2}\Big) \in \Big(\frac{7}{8} + \frac{\sqrt{3}}{768}, \frac{7}{8} + \frac{1}{384}\Big) = \Big(\frac{672 + \sqrt{3}}{768}, \frac{337}{384}\Big).}$$

If instead you used $E_2(\frac{1}{2}) = \frac{\sin(c)}{48}$, you should use

$$0 < \sin(c) \le \sin\Big(\frac{1}{2}\Big) < \sin\Big(\frac{\pi}{6}\Big) = \frac{1}{2},$$

and it follows that $0 \le E_2(\frac{1}{2}) \le \frac{1}{96}$. Then you answer would be the bigger interval

$$\boxed{\cos\Big(\frac{1}{2}\Big) \in \Big(\frac{7}{8}, \frac{7}{8} + \frac{1}{96}\Big) = \Big(\frac{7}{8}, \frac{85}{96}\Big).}$$

At the very least, you should notice that the error is positive so that

$$\boxed{\cos\Big(\frac{1}{2}\Big) \in \Big(\frac{7}{8}, \frac{7}{8} + E\Big),}$$

where E is your estimate from part (b).

5. (11) A large tank contains 10 pounds of salt disolved in 100 gallons of pure water. If water containing 1 pound of salt per gallon is added at a rate of one gallon per minute while the well-mixed solution is drawn off at the same rate, then how many minutes will it take until there is 55 pounds of disolved salt in the tank. (Hint: first find a formula for the number of pounds $A(t)$ of salt in the tank at time t by setting up a first order differential equation and solving it.) *To get full credit you must clearly indicate (a) the differential equation to*

be solved, (b) your solution of the differential equation and (c) your value for the time t required.)

ANS: Since we're adding one pound of salt per minute and drawing off $A(t)/100$ pounds per minute, we need to solve the linear differential equation $\boxed{(a):\ \frac{dA}{dt} = 1 - \frac{1}{100}A}$ with initial condition $\boxed{(a):\ A(0) = 10}$. (Note that the initial condition is part of the answer to part (a).) That is

$$A' + \frac{1}{100}A = 1 \text{ and multiplying by } e^{\frac{t}{100}},$$
$$\left(e^{\frac{t}{100}}A\right)' = e^{\frac{t}{100}} \text{ now we integrate,}$$
$$e^{\frac{t}{100}}A = 100e^{\frac{t}{100}} + C \text{ and}$$
$$A(t) = 100 + Ce^{-\frac{t}{100}}.$$

Since $A(0) = 10$, $C = -90$, and $\boxed{(b):\ A(t) = 100 - 90e^{-\frac{t}{100}}}$.

Now we have to solve for t such that $55 = 100 - 90e^{-\frac{t}{100}}$. Thus $e^{-\frac{t}{100}} = \frac{1}{2}$. Taking log's, gives $-t/100 = \log(1/2) = -\log(2)$. Thus the answer is

$$\boxed{(c):\ t = 100\log(2) \text{ minutes.}}$$

6. (18) MULTIPLE CHOICE. Circle the best response. No partial credit will be given on this problem and you do not need to justify your answers.

(a) The Taylor polynomial of degree 2 for $f(x) = e^{2x}$ about $x = 0$ is

A. $1 + e^{2x}x + \frac{e^{2x}}{x}x^2$ **B**. $1 - 2x^2$ **C**. $1 + 2x + 2x^2$

D. $1 + 2x + 4x^2$ **E**. None of These

ANS: The answer is **C**: just plug $2x$ into the Taylor polynomial for e^x.

(b) The power series $\sum_{n=0}^{\infty}(3x-2)^n$ converges for all x in

A. $(0, 1)$ **B**. $(\frac{1}{3}, 1)$ **C**. $(-\frac{1}{3}, -\frac{1}{3})$ **D**. $(1, 3)$

E. None of These

ANS: Apply the ratio test to get $|3x - 2| < 1$. Thus, the answer is **B**.

(c) The radius of convergence of $\sum_{n=0}^{\infty} 3^n x^{2n}$ is

A. infinite **B**. $\sqrt{3}$ **C**. $\frac{1}{3}$ **D**. $\frac{1}{\sqrt{3}}$ **E**. None of These

ANS: The ratio test implies we want $3|x^2| < 1$. The answer is **D**.

(d) The solution to $\frac{dy}{dx} = y^2 + x$ passing through $(1, -1)$ is most likely to pass closest to which of the following points?

A. $(1.5, -2)$ **B**. $(1.5, -1.5)$ **C**. $(1.5, -3)$ **D**. $(1.5, 0)$

E. $(1.5, -5)$

ANS: Using direction fields, the answer is **D**.

(e) The sum of the series $\frac{3}{2} - \frac{3}{4} + \frac{3}{2^3} - \frac{3}{2^4} + \ldots$ is

A. $\frac{2}{3}$ **B**. $\frac{3}{2}$ **C**. 1 **D**. 2 **E**. None of These

ANS: This is a geometric series with $a = \frac{3}{2}$ and $r = -\frac{1}{2}$. The answer is **C**.

(f) The general solution to $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ is

A. $Ae^{2x} + Be^{-2x}$ **B**. $Ae^{2x} + Bxe^{2x}$ **C**. $Ae^{2x} + Be^{2x}$

D. $Ae^{-2x} + Bxe^{-2x}$ **E**. None of These

ANS: The auxillary equation has only the repeated real root $r = 2$. The answer is **B**.

NAME : ______________________________

Math 8

14 April 2001
Sample Hour Exam I

Instructions:

- Please *print* your name in the space provided.
- A well-prepared student should be able to finish this exam in one hour and fifteen minutes. However, it would be tight.
- Calculators are not allowed. The use of a calculator is a violation of the Honor Code.
- Except on the multiple choice problem (#6), you must show your work and justify assertions to receive full credit.

Problem	Points	Score
1	24	
2	24	
3	7	
4	16	
5	11	
6	18	
Total	100	