Practice Exam 2.
We think this would take a well-prepared student about 75 minutes.

1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

(a) Circle whichever of the following matrix products that are defined:

$$A^2 \quad AB \quad BA \quad B^2$$

ANS: Only BA and B^2 are defined.

(b) Compute the products in part (a) which are defined.

ANS:

$$BA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 2 \end{pmatrix}.$$

$$B^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}.$$

(c) Compute the inverse of the matrix B or else show it has no inverse.

ANS: The matrix is invertible and the inverse is $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

2. An airplane is flying in the jetstream at a speed of 300 miles per hour. The jetstream is blowing due east at a constant speed of 100 miles per hour. The airplane is now 400 miles south of the place where it will begin it's landing path.

(a) What is the tangent of the angle the plane must make with a north-south axis to arrive at the beginning of its landing path?

ANS: We take the positive x axis in the direction of the jetstream and the positive y axis to the north. The velocity vector in the direction of the jetstream is $100\mathbf{i}$. The vector which is the plane's velocity vector is the unknown vector $x\mathbf{i} + y\mathbf{j}$. We want $x\mathbf{i} + y\mathbf{j} + 100\mathbf{i} = c\mathbf{j}$ for some constant c. This gives us $x\mathbf{i} + 100\mathbf{i} = 0$, so $x = -100$. We have $x^2 + y^2 = 300^2$, so $y^2 = 90000 - 10000 = 80000$, and $y = 200\sqrt{2}$. Therefore the tangent of the angle between the plane and the y axis is $\frac{1}{2\sqrt{2}}$.

(b) When will the plane reach the beginning of its landing path?

ANS: $400/200\sqrt{2} = \sqrt{2}$ hours.

3. Find a vector parametric equation for the line of intersection between the planes $x+2y+3z=1$ and $2x+5y+4z=3$.

ANS: We need to solve the system

$$\begin{cases} x + 2y + 3z = 1 \\ 2x + 5y + 4z = 3 \end{cases}$$

which has augmented matrix

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

This corresponds to the equivalent system

$$\begin{cases} x + 7z = -1 \\ y - 2z = 1 \end{cases}$$

Let $z = t$, so that $y = 1 + 2t$ and $x = -1 - 7t$. Thus the parametric equation for the line of intersection is

$$\boxed{\mathbf{r} = (-1 - 7t, 1 + 2t, t).}$$

This is the line through $(-1, 1, 0)$ parallel to the vector $(-7, 2, 1)$.

4. Let A be a 3×4 matrix with row reduced form

$$R = \begin{pmatrix} 1 & 2 & 0 & -4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find all solutions to $A\mathbf{x} = \mathbf{0}$.

ANS: Note that x_2 and x_4 are the free variables. Let $x_2 = t$ and $x_4 = s$. Then we must also have $x_3 = 3s$ and $x_1 = -2t + 4s$. The general solution is then the set of all

$$\begin{aligned} \mathbf{x} = (x_1, x_2, x_3, x_4) &= \boxed{(-2t + 4s, t, 3s, s) \quad \text{for all } r, s \in \mathbb{R}, \text{ or}} \\ &= \boxed{t(-2, 1, 0, 0) + s(4, 0, 3, 1) \quad \text{for all } r, s \in \mathbb{R}} \end{aligned}$$

(b) Let $\mathbf{b}$ be a vector in $\mathbb{R}^3$. Suppose that $\mathbf{x}_p = (1, 1, 0, 1)$ is a solution to

$$A\mathbf{x} = \mathbf{b}.$$

Find the general solution to $A\mathbf{x} = \mathbf{b}$.

ANS: The solution to the nonhomogeneous equation is all vectors of the form $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is any solution to the homogeneous equation in part (a) above. Thus

$$\boxed{\mathbf{x} = (1, 1, 0, 1) + t(-2, 1, 0, 0) + s(4, 0, 3, 1) \quad \text{for all } r, s \in \mathbb{R}}$$

5. Suppose that r and s are real numbers and that

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & r & s \end{pmatrix}.$$

For which values, if any, of r and s does

$$A\mathbf{x} = \mathbf{b}$$

have a solution for all $\mathbf{b}$ in $\mathbb{R}^3$? For what values, if any, of r and s does

$$A\mathbf{x} = \mathbf{b}$$

have a unique solution for all $\mathbf{b}$ in $\mathbf{R}^3$?

ANS: In order that $A\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$, it must be the case that a row reduced form of A have a (nonzero) pivotal entry in each row. This will be the case provided

$$\boxed{\text{at least one of } r \text{ and } s \text{ is different from zero.}}$$

If $r \neq 0$, then a row reduced form of A is

$$R = \begin{pmatrix} 1 & 2 & 0 & 0 & -4 + \frac{2s}{r} \\ 0 & 0 & 1 & 0 & 3 - \frac{2s}{r} \\ 0 & 0 & 0 & 1 & \frac{s}{r} \end{pmatrix}.$$

If $r = 0$, but $s \neq 0$, then a row reduced form of A is

$$R = \begin{pmatrix} 1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In either case, R has three pivotal entries. Of course, if $r = 0 = s$, then a row reduced form of A is

$$R = \begin{pmatrix} 1 & 2 & 0 & -2 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and there aren't enough pivotal entries to guarantee a solution for all $\mathbf{b}$. In no case do we have a unique solution for any b because there are always non-pivotal variables, and they can take on any value.

6. Show that the lines $\mathbf{r}_1 = (2 + t, 4 + 2t, 4 + t)$ and $\mathbf{r}_2 = (2 + s, 3 + 2s, s)$ are parallel and find the equation of the plane which contains both lines.

ANS: Both lines are parallel to the vector $\mathbf{v} = (1, 2, 1)$. Since the first line passes through $(2, 4, 4)$ and second through $(2, 3, 0)$, $\mathbf{w} = (2 - 2, 4 - 3, 4 - 0) = (0, 1, 4)$ is *also parallel* to the plane. Thus we get normal vector by taking the cross-product $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (7, -4, 1)$. Thus the equation of the plane is $7(x - 2) - 4(y - 4) + (z - 4) = 0$, or

$$\boxed{7x - 4y + z = 2.}$$

7. Find the distance from the point $(1, 2, 3)$ to the plane $2x + y - 4z = 5$.

ANS: Let $P_0 = (1, 2, 3)$. Choose a point on the plane, say $Q = (0, 5, 0)$. Let $\mathbf{v}$ be the vector from P_0 to Q, so $\mathbf{v} = ((-1, 3, -3)$. The scalar projection of $\mathbf{v}$ on the normal vector $\mathbf{n} = (2, 1, -4)$ to the plane is given by $\frac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{n}|} = \frac{13}{\sqrt{21}}$. This is the distance from the point $(1, 2, 3)$ to the plane.

8. MULTIPLE CHOICE. Circle the correct response. There will be no partial credit and you do not have to show your work.

(a) Which of the following points does not lie on the line through $(1, 2, 3)$ and parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$?

A. $(3, 1, 2)$ **B**. $(-1, 3, 4)$ **C**. $(2, 1, 1)$ **D**. $(5, 0, 1)$

E. None of These

ANS: The correct answer is C. The line has equation $\mathbf{r} = (1 + 2t, 2 - t, 3 - t)$; just see which points fit and which don't.

(b) Which of the following points does not lie on the plane through $(1, 2, 3)$ with normal vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$?

A. $(0, 0, 3)$ **B**. $(4, -1, 0)$ **C**. $(1, 1, 0)$ **D**. $(0, -9, 0)$

E. None of These

ANS: The correct answer is C. The equation of the plane is $2x - y + 3z = 9$. Now plug in.

(c) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are 3-vectors such that $\mathbf{u} \times \mathbf{v} = 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{u} \bullet \mathbf{u} = 3 = \mathbf{v} \bullet \mathbf{v}$. Then $|\mathbf{u} \bullet \mathbf{v}| =$

A. 0 **B**. 1 **C**. 2 **D**. 3 **E**. None of These

ANS: The correct answer is B. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that $|\mathbf{u} \times \mathbf{v}| = \sqrt{8} = |\mathbf{u}||\mathbf{v}| \sin\theta = 3 \sin\theta$. Thus $\sin\theta = \frac{\sqrt{8}}{3}$, and it follows that $\cos\theta = \pm\frac{1}{3}$. Since $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, we have $\mathbf{u} \bullet \mathbf{v} = \pm 1$.

(d) The scalar projection of $\mathbf{u} = (3, 1, 4)$ in the direction of $\mathbf{v} = (2, 2, 1)$ is

A. 1 **B**. 2 **C**. 3 **D**. 4 **E**. None of These

ANS: The correct answer is D. Here $s := \frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = \frac{12}{3} = 4$.

(e) Which of the following is *not* perpendicular to $\mathbf{i} + \mathbf{j} + \mathbf{k}$?

A. $3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ **B**. $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ **C**. $\mathbf{j} - \mathbf{k}$ **D**. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

E. None of These

ANS: The correct answer is E. The dot product with the given vectors are all zero.

(f) Find the cosine of the angle between the planes $x + 2y + z = 9$ and $-x - y + 2z = 5$. (Note that the angle between the planes is the same as the angle between the normal vectors to the planes.)

A. $-\frac{\pi}{4}$ **B**. $\frac{5\pi}{7}$ **C**. $-\frac{1}{6}$ **D**. $\frac{5}{9}$ **E**. None of These

ANS: The correct answer is C. Find the angle between the normal vectors:
$\cos\theta = \dfrac{(1, 2, 1) \bullet (-1, -1, 2)}{\sqrt{6}\sqrt{6}} = -\dfrac{1}{6}$.

(g) If $A = \begin{pmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, number of parameters in the most general solution to $A\mathbf{x} = \mathbf{0}$ is

A. 1 **B**. 2 **C**. 3 **D**. 4 **E**. 5

ANS: The correct answer is B: the dimension equals the number of free variables.

(h) The area of the triangle with vertices $P_1(1,0,1)$, $P_2(2,2,2)$ and $P_3(3,4,5)$ is

A. 3 **B**. $\sqrt{6}$ **C**. 4 **D**. $\sqrt{3}$ **E**. $\sqrt{5}$

ANS: The correct answer is E. The vectors $\mathbf{v} = (1,2,1)$ from P_1 to P_2 and $\mathbf{w} = (2,4,4)$ from P_1 to P_3 span a parallelogram of area $|\mathbf{v} \times \mathbf{w}| = |(4,-2,0)| = 2\sqrt{5}$. The area of the triangle is half the area of the parallelogram, i.e., $\sqrt{5}$.