1. (a) Find and classify any critical points of $f(x,y) = x^2 + x + 2y^2$ in the region $x^2 + y^2 < 1$.

ANS: $\dfrac{\partial f(x,y)}{\partial x} = 2x + 1$; $\dfrac{\partial f(x,y)}{\partial y} = 4y$. Therefore the only critical point is where $2x+1=0$ and $4y=0$, so $x=-1/2$ and $y=0$. At this point $f(x,y) = \frac{1}{4} - \frac{1}{2} + 0 = -\frac{1}{4}$ (to be used later). We have $A=2$, $B=0$, and $C=4$, so $B^2 < AC$ and $A > 0$ so the critical point is a local minimum

ANS: $(-\frac{1}{2}, 0)$, local minimum.

(b) Find the extreme values of $f(x,y) = x^2 + x + 2y^2$ on the region $x^2 + y^2 \le 1$.

ANS: Setting $\nabla f(x,y) = \lambda \nabla (x^2 + y^2)$, we obtain $(2x+1) = \lambda 2x$ and $4y = \lambda 2y$. The constraint is $x^2 + y^2 = 1$. We thus have $(2\lambda - 2)x = 1$ and $(4 - 2\lambda)y = 0$. The second equation says that either $y = 0$ or else $\lambda = 2$. If $y = 0$, then $x = \pm 1$, by the equation of the circle. If $\lambda = 2$, then $x = \frac{1}{2}$, and by the equation of the circle, $y = \pm\frac{\sqrt{3}}{2}$. Our constrained critical points are thus $(\pm 1, 0)$, $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. We have $f(1,0) = 2$, $f(-1,0) = 0$ and $f(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{9}{4}$. We compare these values with the value $\frac{-1}{4}$ at the critical point found in part (a).

The maximum is 9/4 which occurs at the points $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$.

The minimum is -1/4 which occurs at the points $(-\frac{1}{2}, 0)$.

2. (a) Find the general solution to $y'' + 4y' + 4y = 0$.

ANS: The auxilliary equation is $r^2 + 4r + 4 = (r+2)^2 = 0$, so $r = -2$, and therefore $y = C_1 e^{-2x} + C_2 x e^{-2x}$.

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}.$$

(b) Given that $y_p(x) = x$ is a solution to the nonhomogeneous equation

$$(\dagger) \qquad y'' + 4y' + 4y = f(x),$$

find the solution to (†) which satisfies $y(0) = 2$ and $y'(0) = 1$.

ANS: $y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + x$, so $y' = -2C_1 e^{-2x} - 2xC_2 e^{-2x} + C_2 e^{-2x} + 1$. Substituting in the initial conditions gives us $C_1 = 2$ and $-2C_1 + C_2 = 0$, which gives us $C_2 = 4$.

$$y(x) = 2e^{-2x} + 4xe^{-2x} + x \ .$$

(c) What is $f(x)$ equal to?

ANS: Substituting $y = x$ into Equation (†) gives $4x + 4$.

$$f(x) = 4x + 4 \ .$$

3. Suppose that $z = f(x, y)$, $x = uv$ and $y = u + 3v$. When $u = 2$ and $v = 1$, then $\dfrac{\partial z}{\partial u} = -2$ and $\dfrac{\partial z}{\partial v} = -1$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

ANS: $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} = -2$, or $\dfrac{\partial z}{\partial x}v + \dfrac{\partial z}{\partial y}\cdot 1 = -2$, and since $v = 1$, this gives $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = -2$. Similarly $\dfrac{\partial z}{\partial x}\cdot 2 + \dfrac{\partial z}{\partial y}\cdot 3 = -1$. Solving these equations gives

$$\underline{\frac{\partial z}{\partial x} = -5} \ .$$

$$\underline{\frac{\partial z}{\partial y} = 3} \ .$$

4. (a) Find the equation of the plane containing the three points $(1, 1, 1)$, $(2, 3, 2)$ and $(2, 0, 3)$.

ANS: To get a normal to the plane, we take the cross product of $(2, 3, 2) - (1, 1, 1) = (1, 2, 1)$ and $(2, 0, 3) - (1, 1, 1) = (1, -1, 2)$.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}.$$

The equation of the plane is

$$(x - 1, y - 1, z - 1) \cdot (5, -1, -3) = 5(x - 1) - 1(y - 1) - 3(z - 1) = 0,$$

or

ANS: $\underline{5x - y - 3z = 1}$

(b) Find parametric equations for the line of intersection of the planes $x + 2y + 2z = 1$ and $-x + y - 8z = 2$.

ANS:
$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ -1 & 1 & -8 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 3 & -6 & 3 \\ -1 & 1 & -8 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ 1 & -1 & 8 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -2 & 1 \\ 1 & 0 & 6 & -1 \end{array}\right]$, so $x = -1 - 6t$, $y = 1 + 2t$, and $z = t$.

$\underline{x = -1 - 6t}$

$\underline{y = 1 + 2t}$

$\underline{z = t}$

5. Find all solutions to the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 - \ x_4 &= 1 \\ x_1 \qquad - x_3 + 2x_4 &= 1 \\ x_2 + x_3 + \ x_4 &= 3 \\ 2x_1 \qquad + x_3 + 2x_4 &= 5 \end{aligned}$$

What kind of geometric object in four-dimensional space is the set of all solutions to the system of equations? (Be as specific as you can, but be brief!)

ANS: $\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 0 & 1 & 2 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 2 & -1 & 4 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & 2 & -3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & 2 & -3 \end{array}\right] \longrightarrow$

$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & -3 & 2 & -3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & \frac{-2}{3} & 1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{4}{3} & 2 \\ 0 & 0 & 1 & \frac{-2}{3} & 1 \\ 0 & 1 & 0 & \frac{5}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Therefore, $x_1 = 2 - \frac{4}{3}t$, $x_2 = 2 - \frac{5}{3}t$, $x_3 = 1 + \frac{2}{3}t$, $x_4 = t$ is the general solution to the system of equations. The set of all solutions is a line, the line through the point (2,2,1,0) with direction vector $(-\frac{4}{3}, -\frac{5}{3}, \frac{2}{3}, 1)$.

6. Find the *minimum* of $f(x, y, z) = x^2 + y^2 + 2z^2$ on the plane $x + y + z = 2$.

ANS: Setting $\nabla(x^2 + y^2 + 2z^2) = \lambda\nabla(x + y + z)$ gives $2x = \lambda$, $2y = \lambda$, $4z = \lambda$. Thus $x = y$ and $z = y/2$. Since $x + y + z = 2$, this gives us $x = 4/5$, $y = 4/5$, $z = 2/5$, at which point $x^2 + y^2 + 2z^2 = 8/5$. Since f is large for values of x, y and z such that at least one is large in absolute value, f must have a minimum somewhere on this plane. Therefore the point we have found is the only possible point at which this minimum can occur.

The minimun is $\underline{8/5}$ which occurs at $\underline{(4/5, 4/5, 2/5)}$.

7. The temperature at the point (x, y, z) is given by $T(x, y, z) = xy^2 + yz^2$. You are located at $(1, 1, 1)$.

(a) If you move in the direction $\mathbf{u} = \frac{1}{\sqrt{3}}(1, -1, 1)$, then does your temperature increase, decrease or stay the same? Explain.

ANS: $\nabla T = y^2\mathbf{i} + (2xy + z^2)\mathbf{j} + 2yz\mathbf{k} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ at (1,1,1). Since $(1, 3, 2) \cdot (1, -1, 1)/\sqrt{3} = 0$, the temperature does not change.

ANS: Stays the same.

(b) In what direction should you head so that your temperature decreases as rapidly as possible?

ANS: $-\nabla T = -\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

ANS: $(-\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})/\sqrt{14}$. (A unit vector is not really necessary here.)

(c) What is your maximum rate of cooling in part (b)?

ANS: $\sqrt{14}$.

8. MULTIPLE CHOICE. Circle the correct response. There will be no partial credit and you do not have to show your work.

(a) The MacLaurin Series for $f(x) = e^x$ is

A. $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$ **B**. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ **C**. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

D. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n!}$ **E**. None of These

ANS: B

(b) The MacLaurin Series for $f(x) = \cos x$ is

A. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}$ **B**. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ **C**. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

D. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n!}$ **E**. None of These

ANS: C

(c) The MacLaurin Series for $f(x) = xe^{-x}$ is

A. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}$ **B**. $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ **C**. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

D. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n!}$ **E**. None of These

ANS: D

(d) Suppose that P is parallelogram with vertices at $(1,1,1)$, $(2,3,4)$ and $(2,4,3)$. The fourth vertex is

A. $(0,0,0)$ **B**. $(3,6,6)$ **C**. $(3,6,7)$ **D**. $(4,5,6)$

E. None of These

ANS: B

(e) The function $f(x,y) = 2x^3 - 6xy + 3y^2$ has a critical point at $(1,1)$. At $(1,1)$, f has a

A. local minimum **B**. local maximum **C**. saddle point

D. singularity **E**. None of These

ANS: A

(f) The tangent plane to the surface $z = x^2y + xy^2$ at the point $(1,2,6)$ is

A. $8x + 5y = 12$ **B**. $x + 2y - z = 6$ **C**. $8x + 5y - z = 12$

D. $2x + y + z = 6$ **E**. None of These

ANS: C

(g) Suppose that $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $y(0) = 2$. Then $y(1) =$

A. 0 **B**. 1 **C**. $\sqrt{2}$ **D**. $\sqrt{3}$ **E**. 2

ANS: D

(h) The general solution to $y' + 3y = e^x$ is

A. $\frac{1}{4} + Ce^{3x}$ **B**. $\frac{1}{4}e^x + C$ **C**. $\frac{1}{4} + Ce^{-3x}$ **D**. $\frac{1}{4}e^x + Ce^{-3x}$

E. None of These

ANS: D

(i) The radius of convergence of the power series $\sum_{n=0}^{\infty} 2^n x^n$ is

A. 0 **B**. 2 **C**. $\frac{1}{2}$ **D**. ∞ **E**. None of These

ANS: C

(j) Suppose that $A = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$. Then the number of parameters in a general solution for equation $A\mathbf{x} = \mathbf{0}$ is

A. 1 **B**. 2 **C**. 3 **D**. 4 **E**. None of These

ANS: B