SOLVING LINEAR HOMOGENEOUS SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In these notes, we will consider differential equations of the form

$$ay'' + by' + cy = 0$$

where a, b and c are constants.

Notation. We can obtain surprising insight into the differential equation by using the notation $\mathbf{D}y$ to mean y' and $\mathbf{D}^2y$ to mean y''. In other words, $\mathbf{D}$ denotes the derivative operator $\frac{d}{dx}$ and is called a differential operator. We will write expressions such as $(5\mathbf{D}^2+2\mathbf{D}+7)y$ to mean $5\mathbf{D}^2y+2\mathbf{D}y+7y = 5y''+2y'+7y$.

How does this new notation help us understand the differential equation. Let's look at an example.

Example 1..

$$y'' - 3y' + 2y = 0. \tag{1}$$

Writing the equation in the form

$$(\mathbf{D}^2 - 3\mathbf{D} + 2)y = 0,$$

we find ourselves tempted to factor the expression $\mathbf{D}^2 - 3\mathbf{D} + 2$, writing the equation as

$$(\mathbf{D} - 2)(\mathbf{D} - 1)y = 0. \tag{2}$$

Does this really make sense? Let's see. First, we know that

$$(\mathbf{D} - 1)y = y' - y. \tag{3}$$

Thus

$$(\mathbf{D} - 2)(\mathbf{D} - 1)y = (\mathbf{D} - 2)(y' - y) = \mathbf{D}(y' - y) - 2(y' - y). \tag{4}$$

Since $\mathbf{D}(y' - y)$ means differentiate $y' - y$, we see that $\mathbf{D}(y' - y) = y'' - y'$. Thus equation (4) gives us

$$(\mathbf{D} - 2)(\mathbf{D} - 1)y = y'' - y' - 2(y' - y) = y'' - 3y' + 2y$$

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in agreement with the original expression in (1). Thus factoring the differential operator $\mathbf{D}^2 - 3\mathbf{D} + 2$ does make sense.

From equation (2), we can see that any solution $y(t)$ of the differential equation

$$(\mathbf{D} - 1)y = 0 \tag{5}$$

must also be a solution of equation (2), equivalently equation (1). But we know how to solve equation (5) already. Rewriting this equation as $y' - y = 0$ or $y' = y$, the solution is

$$y = C_1 e^t \tag{6}$$

where C_1 is an arbitrary constant.

Now we could have just as easily rewritten our differential equation (1) in the form

$$(\mathbf{D} - 1)(\mathbf{D} - 2)y = 0. \tag{7}$$

From this expression, we immediately see that any solution $y(t)$ of $(\mathbf{D}-2)y = 0$ is also a solution of (1). Again this equation can be rewritten as $y' - 2y = 0$, or $y' = 2y$, giving us the solution

$$y = C_2 e^{2t} \tag{8}$$

Since the differential equation is homogeneous, we can combine these solutions to obtain solutions of the form

$$y = C_1 e^t + C_2 e^{2t}. \tag{9}$$

What is the advantage of viewing the differential equation in this way? After all, we already know another way of solving this equation by guessing a solution of the form e^{rt} and finding r. Let's consider the following example:

Example 2. Consider the differential equation $y'' - 4y' + 4y = 0$. If we guess a solution of the form $y = e^{rt}$, we find that $r^2 - 4r + 4 = 0$, so $(r - 2)^2 = 0$. This gives us only the solution $y = e^{rt}$. We can multiply this solution by an arbitrary constant, but we're still missing a second independent solution. Now let's write our equation in the form $(\mathbf{D}^2 - 4\mathbf{D} + 4)y = 0$ or

$$(\mathbf{D} - 2)^2 y = 0. \tag{10}$$

We seem to run into the same problem. We see that any solution of $(\mathbf{D}-2)y = 0$ is a solution to our original equation, but this still just gives us $y = C_1 e^{rt}$. How do we find another solution?

Let's consider a more familiar equation: $y'' = 0$, i.e., $\mathbf{D}^2 y = 0$. If we let $v = y'$ $(= \mathbf{D}y)$, then we must have $v' = 0$ and thus $v = C_1$, for some constant

C_1. Thus $y' = C_1$, so $y = C_1 t + C_2$. Note what we've done here. We've replaced $\mathbf{D}y$ by v, thus rewriting $\mathbf{D}^2 y = 0$ as a first order equation $\mathbf{D}v = 0$. Once we found v, we could then solve $\mathbf{D}y = v$ for y. Let's do the same thing with equation (10). Let

$$v = (\mathbf{D} - 2)y. \tag{11}$$

Equation (10) tells us that

$$(\mathbf{D} - 2)v = 0.$$

We know how to solve this equation: $v' - 2v = 0$, so $v' = 2v$ and

$$v = C_1 e^{2t}.$$

Substituting our solution for v into equation (11), we obtain

$$(\mathbf{D} - 2)y = C_1 e^{2t}$$

or equivalently

$$y' - 2y = C_1 e^{2t}. \tag{12}$$

Let's solve this linear first order equation by using an integrating factor. We choose u to be an anti-derivative of -2, say $u = -2t$ and multiply both sides by e^{-2t} to obtain

$$\frac{d}{dt}(e^{-2t} y) = C_1.$$

(On the right-hand-side, we've replaced $C_1 e^{2t} e^{-2t}$ by C_1.) Thus

$$e^{-2t} y = C_1 t + C_2$$

and

$$y = C_1 t e^{2t} + C_2 e^{2t}.$$

We have now found a two-dimensional space of solutions. Moreover, we know these are *all* the solutions since at each stage, we solved a first order equation completely, finding all its solutions.