Mathematics 8
Problems for Exam 1

The following problems were considered for the exam, but ultimately not included. This document is not indicative of the length or the distribution of problems on the actual exam.

1. A particle moving along the x-axis has position $x = 0$ at time $t = 0$, and at time t has velocity $v(t) = \dfrac{1}{\sqrt{1+t^2}}$.

   (a) Find the position of the particle at time t, for $t \geq 0$.

   Solution:
$$\int_0^t v(u)\,du = \int_0^t \frac{1}{\sqrt{u^2+1}}\,du$$
   Use a trigonometric substitution with $u = \tan\theta$, $du = \sec^2\theta\,d\theta, 0 \leq \theta < \dfrac{\pi}{2}$. Note that in this range $\sec\theta$ is positive, so $\sqrt{u^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = \sec\theta$.
$$\int \frac{1}{\sqrt{u^2+1}}\,du = \int \frac{1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta = \int \sec\theta\,d\theta =$$
$$\ln(\sec\theta + \tan\theta) + C = \ln(\sqrt{1+u^2}+u) + C$$
$$\int_0^t \frac{1}{\sqrt{u^2+1}}\,du = \ln(\sqrt{1+t^2}+t) - \ln(\sqrt{1+0^2}+0) = \ln(\sqrt{1+t^2}+t).$$

   (b) Find the average acceleration of the particle between times $t = 0$ and $t = 10$.

   Solution: The acceleration at time t is $v'(t)$, so the average acceleration between $t = 0$ and $t = 10$ is
$$\frac{1}{10-0}\int_0^{10} v'(t)\,dt = \frac{1}{10}\left(v(10) - v(0)\right) = \frac{1}{10}\left(\frac{1}{\sqrt{101}} - 1\right).$$

2. Find the volume of the solid obtained by rotating the area under the curve $y = \sin x$, for $0 \leq x \leq \pi$, about the x-axis.

Solution: The cross-sectional area at x is the area of a circle with radius $\sin x$, or $A(x) = \pi \sin^2 x$. Therefore the volume is

$$\int_0^\pi \pi \sin^2 x \, dx.$$

Using the half-angle formulas,

$$\int_0^\pi \pi \sin^2 x \, dx = \pi \int_0^\pi \frac{1 - \cos 2x}{2} \, dx = \pi \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \Bigg|_0^\pi = \frac{\pi^2}{2}.$$

3. Evaluate the following integrals

(a) $\displaystyle\int_0^\infty e^{-x}\sin x\ dx$

Solution: First evaluate the indefinite integral $\int e^{-x}\sin x\ dx$ using two applications of integration by parts:

$$\begin{aligned} u &= e^{-x} & dv &= \sin x\ dx \\ du &= -e^{-x}dx & v &= -\cos x \end{aligned}$$

Then

$$\int e^{-x}\sin x\ dx = -e^{-x}\cos x - \int e^{-x}\cos x\ dx$$

The second integration by parts:

$$\begin{aligned} u &= e^{-x} & dv &= \cos x\ dx \\ du &= -e^{-x}dx & v &= \sin x \end{aligned}$$

Then

$$\begin{aligned} \int e^{-x}\sin x\ dx &= -e^{-x}\cos x - \left(e^{-x}\sin x + \int e^{-x}\sin x\ dx\right) \\ &= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\ dx \end{aligned}$$

The integral on each side of the equation is the same so we can add it to both sides

$$2\int e^{-x}\sin x\ dx = -e^{-x}(\cos x + \sin x)$$

The final indefinite integral is

$$\int e^{-x}\sin x\ dx = \frac{-e^{-x}}{2}(\cos x + \sin x) + C$$

Now we can use the definition of improper integral to get the final answer

$$\begin{aligned} \int_0^\infty e^{-x}\sin x\ dx &= \lim_{t\to\infty}\int_0^t e^{-x}\sin x\ dx \\ &= \lim_{t\to\infty}\left[\frac{-e^{-x}}{2}(\cos x + \sin x)\right]_0^t \\ &= \lim_{t\to\infty}\left[\frac{-e^{-t}}{2}(\cos t + \sin t) - \frac{-1}{2}\right] \end{aligned}$$

Finally to evaluate the limit we can use the squeeze theorem. Let $f(t) = -1/e^t$, $g(t) = (\cos t)/e^t$ and $h(t) = 1/e^t$. Then $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} h(t) = 0$ and $f(t) \le g(t) \le h(t)$ for $t \ge 0$. So by the squeeze theorem (cf. section 1.6 in Stewart) we can conclude that $\lim_{t\to\infty} g(t) = 0$. Exactly the same reasoning allows use to conclude that $\lim_{t\to\infty}(\sin t)/e^t = 0$. Hence the limit above becomes

$$\begin{aligned}
\int_0^\infty e^{-x}\sin x\,dx &= \lim_{t\to\infty}\left[\frac{-e^{-t}}{2}(\cos t+\sin t) - \frac{-1}{2}\right]\\
&= \frac{-1}{2}\left(\lim_{t\to\infty}\frac{\cos t}{e^t} + \lim_{t\to\infty}\frac{\sin t}{e^t}\right) + \frac{1}{2}\\
&= \frac{-1}{2}(0+0) + \frac{1}{2}\\
&= \frac{1}{2}
\end{aligned}$$

(b) $\displaystyle\int_0^4 \frac{dx}{(9+x^2)^{3/2}}$

Solution: Use trigonometric substitution with $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\,d\theta$ and the lower and upper bounds of the integration become 0 and $\tan^{-1}(4/3)$ respectively.

$$\begin{aligned}
\int_0^4 \frac{dx}{(9+x^2)^{3/2}} &= \int_0^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{(9+9\tan^2\theta)^{3/2}}\\
&= \int_0^{\tan^{-1}(4/3)} \frac{3\sec^2\theta}{9^{3/2}(\sec\theta)^3}d\theta\\
&= \frac{3}{27}\int_0^{\tan^{-1}(4/3)} \frac{d\theta}{\sec\theta}\\
&= \frac{1}{9}\int_0^{\tan^{-1}(4/3)} \cos\theta\ d\theta\\
&= \frac{1}{9}\Big[\sin\theta\Big]_0^{\tan^{-1}(4/3)}\\
&= \frac{1}{9}\Big[\sin(\tan^{-1}(4/3)) - \sin(0)\Big]
\end{aligned}$$

If the sides of a right triangle have length 3 and 4, then the hypotenuse has length $\sqrt{3^2+4^2} = \sqrt{25} = 5$. So the first term in brackets becomes $4/5$ and the integral evaluates to

$$\int_0^4 \frac{dx}{(9+x^2)^{3/2}} = \frac{1}{9}\left[\frac{4}{5} - 0\right] = \frac{4}{45}$$

(c) $\displaystyle\int_0^{\sqrt{5}} \frac{x^3}{\sqrt{x^2+4}}dx$

Solution: Use trigonometric substitution with $x = 2\tan\theta$. Then $dx = 2\sec^2\theta d\theta$ and the lower and upper bounds of integration become 0 and $\tan^{-1}(\sqrt{5}/2)$ respectively. Substituting into the integral

$$\begin{aligned}
\int_0^{\sqrt{5}} \frac{x^3}{\sqrt{x^2+4}}dx &= \int_0^{\tan^{-1}(\sqrt{5}/2)} \frac{8\tan^3\theta\ 2\sec^2\theta}{\sqrt{4\tan^2\theta+4}}d\theta \\
&= \int_0^{\tan^{-1}(\sqrt{5}/2)} \frac{8\tan^3\theta\ 2\sec^2\theta}{2\sqrt{\sec^2\theta}}d\theta \\
&= 8\int_0^{\tan^{-1}(\sqrt{5}/2)} \tan^3\theta\ \sec\theta\ d\theta \\
&= 8\int_0^{\tan^{-1}(\sqrt{5}/2)} \tan^2\theta\ (\sec\theta\tan\theta)\ d\theta \\
&= 8\int_0^{\tan^{-1}(\sqrt{5}/2)} (\sec^2\theta - 1)(\sec\theta\tan\theta)\ d\theta
\end{aligned}$$

Now we can use a u substitution with $u = \sec\theta$. Then $du = \sec\theta\tan\theta\ d\theta$ and the lower and upper bounds of integration become 1 and $3/2$ respectively. So

$$\begin{aligned}
\int_0^{\sqrt{5}} \frac{x^3}{\sqrt{x^2+4}}dx &= 8\int_1^{3/2} (u^2-1)du \\
&= 8\left[\frac{u^3}{3} - u\right]_1^{3/2} \\
&= 8\left[\frac{1}{3}\cdot\frac{27}{8} - \frac{3}{2} - \left(\frac{1}{3} - 1\right)\right] = 8\frac{7}{24} = \frac{7}{3}
\end{aligned}$$

Note that you could also have used integration by parts with $u = x^2$ and $dv = (x/\sqrt{x^2+4})dx$.

4. Determine whether the following converge or diverge. Be sure to explain your reasoning.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n - 3}{n}$.

   Solution: We can use the comparison test here. For $n > e^4$, we have $\ln n > 4$, $\ln n - 3 > 1$, and $\dfrac{\ln n - 3}{n} > \dfrac{1}{n}$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (we know this because it is a p series with $p \geq 1$), by the comparison test this series also diverges.

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{\ln(n^2)}$

   Solution: Checking the divergence test the limit of the sequence of terms is

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(n^2)} = \lim_{n\to\infty} \frac{\ln(n)}{2\ln(n)} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

   Since the limit is not zero, the divergence test allows us to conclude that the series diverges.

   (c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

   Solution: A quick check shows that the sequence of the terms converges to zero, so the divergence test does not apply. Near $x = 0$ the function $\ln(1+x)$ behaves roughly like x (check the slope of the tangent line at $x = 0$ to confirm this). So the composition $\ln(1 + 1/n)$ behaves roughly like $1/n$ as $n \to \infty$. This suggests that we should compare this series to the harmonic series $\sum b_n = \sum 1/n$. Let $a_n = \ln(1 + 1/n)$ and $f(x) = \ln(1 + 1/x)$ and $g(x) = 1/x$. Then f and g are continuous on $[1, \infty)$. Since the f and g converge to zero as $x \to \infty$ and the derivative of g is not zero on $[1, \infty)$, we can use l'Hôpital's rule to conclude

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \\ &= \lim_{x\to\infty} \frac{(\frac{1}{1+1/x})(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1\end{aligned}$$

   Since both a_n and b_n are positive for all n and their ratio converges to a positive number, the limit comparison test allows us to conclude that both series either diverge or converge together. Since the harmonic series diverges, the given series $\sum \ln(1 + 1/x)$ also diverges.

5. Evaluate the following. (Your answer should be a number, $+\infty$, $-\infty$, or "diverges" if it diverges but not to $+\infty$ or $-\infty$.) Be sure to explain your reasoning.

(a) $\lim\limits_{n\to\infty} \dfrac{\ln(n^3+5)}{n}$

Solution: $\lim\limits_{n\to\infty} \dfrac{\ln(n^3+5)}{n} = \lim\limits_{x\to\infty} \dfrac{\ln(x^3+5)}{x}$ (assuming this limit exists), which we can evaluate using l'Hospital's Rule:

$$\lim_{x\to\infty} \frac{\ln(x^3+5)}{x} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3+5}}{1} = \lim_{x\to\infty} \frac{3x^2}{x^3+5} = \lim_{x\to\infty} \frac{6x}{3x^2} = \lim_{x\to\infty} \frac{6}{6x} = 0$$

(b) $\lim\limits_{n\to\infty} \dfrac{n\ln(n)}{n^2+5}$

Solution: Let $a_n = n\ln(n)/(n^2+5)$ and let function $f(x) = x\ln(x)/(x^2+5)$. The function f is continuous on $[1,\infty)$ and $a_n = f(n)$, so

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

Since both the numerator and denominator of $f(x)$ diverge to infinity and the derivative of the denominator is not zero on $[1,\infty)$, we can use l'Hôpital's rule

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(x) + x(1/x)}{2x} = \lim_{x\to\infty} \frac{\ln(x)+1}{2x}$$

Again we are in a situation where we can apply l'Hôpital's rule and

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{\ln(x)+1}{2x} = \lim_{x\to\infty} \frac{1/x}{2} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

(c) $\lim\limits_{n\to\infty} n^2(\cos(1/n) - 1)$

Solution: Note that the first factor n^2 diverges to infinity, while the second factor in parentheses converges to zero. This indicates that we might be able to use l'Hôpital's rule after rewriting it as a fraction as follows. Let

$$a_n = n^2(\cos(1/n) - 1) = \frac{\cos(1/n) - 1}{1/n^2}$$

Let $f(x) = x^2(\cos(1/x) - 1)$. Then f is continuous on $[1,\infty)$ and checking that the derivative of the denominators is not zero, we can apply l'Hôpital's rule twice to get

$$\begin{aligned}\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{\cos(1/x) - 1}{1/x^2} = \lim_{x\to\infty} \frac{-\sin(1/x)(-1/x^2)}{-2/x^3} \\ &= \lim_{x\to\infty} \frac{-\sin(1/x)}{2/x} = \lim_{x\to\infty} \frac{-\cos(1/x)(-1/x^2)}{-2/x^2} = \lim_{x\to\infty} \frac{-1}{2}\cos(1/x) = \frac{-1}{2}\end{aligned}$$

6. Let $a_n = \dfrac{1}{(n+3)^3}$.

Assume you want to use the partial sum $s_c = \sum_{n=1}^{c} a_n$ to approximate the value of $\sum_{n=1}^{\infty} a_n$.

To guarantee an error of at most $\dfrac{8}{10^6}$, how large must c be?

Solution: Since $f(x) = \dfrac{1}{(x+3)^3}$ is continuous and decreasing for $x \geq 0$, we can use the error estimate from the integral test. The remainder, or error, term is

$$R_c = \sum_{n=c+1}^{\infty} \frac{1}{(n+3)^3} \leq \int_c^{\infty} \frac{1}{(x+3)^3}\,dx = \lim_{b\to\infty} \int_c^b \frac{1}{(x+3)^3}\,dx =$$

$$= \lim_{b\to\infty} \left(\frac{-1}{2(b+3)^2} - \frac{-1}{2(c+3)^2} \right) = \frac{1}{2(c+3)^2}.$$

We want

$$\frac{1}{2(c+3)^2} \leq \frac{8}{10^6} \qquad \frac{1}{(c+3)} \leq \frac{4}{10^3} \qquad (c+3) \geq \frac{10^3}{4} \qquad c \geq 247.$$