1. Consider the function
$$f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^2}.$$
For which values of x is $f(x)$ defined?

Solution: The function $f(x)$ is defined at values of x for which the series converges.

We find the radius of convergence using the ratio test.
$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)}{(n+2)^2}}{\frac{(x-3)^n}{(n+1)^2}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2(x-3)}{(n+2)^2} \right| = |x-3| \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+2)^2} \right| = |x-3|.$$
The ratio test tells us that the series converges when $|x-3| < 1$ and diverges when $|x-3| > 1$.

When $|x-3| = 1$ the ratio test fails, and we have to try something else. For $x-3=1$, or $x=4$, we have
$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2},$$
which converges because it is a p-series with $p=2$. For $x-3=-1$, or $x=2$, we have
$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2},$$
which converges by the alternating series test. It also converges because it converges absolutely.

Therefore the series converges, and $f(x)$ is defined, when $|x-3| \le 1$, or when $2 \le x \le 4$.

2. (a) What is the Maclaurin series (the Taylor series about $x = 0$) for the function $f(x) = e^x$?

You do not need to show any work for this part of the problem, so if you remember the answer, you can just write it down.

Solution:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) Find the Maclaurin series for $g(x) = e^{-x^2}$.

Solution: Substituting $-x^2$ for x in the Maclaurin series for e^x, we get

$$\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

(c) Let $F(x) = \displaystyle\int_0^x e^{-t^2}\, dt$. Find the first four nonzero terms of the Maclaurin series for $F(x)$.

Solution: Replacing x by t in the Maclaurin series for e^{-x^2} yields the Macluarin series for the integrand

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Integrating this power series term-by-term

$$\int e^{t^2}\, dt = C + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)n!}$$

Evaluating the integral from $t = 0$ to $t = x$ gives

$$\begin{aligned}\int_0^x e^{t^2}\, dt &= \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)n!}\right]\Bigg|_{t=0}^{t=x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \\ &= x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\end{aligned}$$

So the first four nonzero terms are

$$x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!}$$

3. Find the Taylor series for $f(x) = \ln(x) - \ln(3)$ centered about $a = 3$.

Solution: Calculating the first few derivatives of $f(x)$, we have

$$\begin{aligned}
f^{(0)}(x) &= \ln(x) - \ln(3) \\
f^{(1)}(x) &= 1/x \\
f^{(2)}(x) &= -1/x^2 \\
f^{(3)}(x) &= 2/x^3 \\
f^{(4)}(x) &= -2 \cdot 3/x^4 \\
f^{(5)}(x) &= 2 \cdot 3 \cdot 4/x^4
\end{aligned}$$

we find that if $n \geq 1$, then the n-th derivative is $f^{(n)}(x) = (-1)^{n+1}(n-1)!/x^n$. Hence Taylor's formula gives

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(3) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{3^n n!}(x-3)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}(x-3)^n
\end{aligned}$$

4. Suppose that $\vec{a}$ and $\vec{b}$ are two vectors in $\mathbb{R}^3$ such that if $\vec{a}$ and $\vec{b}$ are drawn emanating from the origin they both lie in the xy-plane, $\vec{a}$ in the third quadrant ($x < 0$ and $y < 0$) and $\vec{b}$ in the second quadrant ($x < 0$ and $y > 0$).

   Suppose also that we know $|\vec{a}| = 1$ and $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

   (a) Is the projection of $\vec{b}$ onto $\vec{a}$ longer than $\vec{a}$, shorter than $\vec{a}$, or the same length as $\vec{a}$?

   Solution:

   If θ is the angle between $\vec{a}$ and $\vec{b}$, then

   $$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = 1 \qquad |\vec{b}| = \sqrt{\vec{b} \cdot \vec{b}} = 2 \qquad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{2}$$

   The length of the projection of $\vec{b}$ onto $\vec{a}$ is $|\vec{b}|\,|\cos\theta| = 1$, so it is the same length as $\vec{a}$.

   (b) In what direction does $\vec{a} \times \vec{b}$ point?

   Solution:

   It must point in a direction normal to both $\vec{a}$ and $\vec{b}$, that is, normal to the xy-plane, so either the direction given by $\hat{k}$ (the positive z direction) or the direction given by $-\hat{k}$ (the negative z direction). Looking down from the top ($\hat{k}$, or positive z, direction) of the xy-plane we see that from $\vec{a}$ to $\vec{b}$ is a clockwise direction, so by the right-hand rule, the cross product points in the direction given by $-\hat{k}$.

   (c) Find the length of $\vec{a} \times \vec{b}$.

   Solution:

   If $\cos\theta = \dfrac{1}{2}$ then $\sin\theta = \dfrac{\sqrt{3}}{2}$ (we know it cannot be negative because we always take θ to be an acute angle) so

   $$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta = \boxed{\sqrt{3}}$$

5. Find an equation for the plane that contains the two lines L_1 and L_2:

$$L_1: \quad x = t + 2 \qquad y = 3t - 5 \qquad z = 5t + 1$$

$$L_2: \quad x = 5 - t \qquad y = 3t - 10 \qquad z = 9 - 2t$$

Solution: A vector that points in the same direction as L_1 is $\mathbf{v_1} = \langle 1, 3, 5\rangle$ and one that points in the same direction as L_2 is $\mathbf{v_2} = \langle -1, 3, -2\rangle$. Since the plane contains both lines, a normal vector to the plane must be orthogonal to $\mathbf{v_1}$ and $\mathbf{v_2}$. We can construct a normal vector using the cross product:

$$\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 5 \\ -1 & 3 & -2 \end{bmatrix} = (-6 - 15)\hat{\imath} - (-2 + 5)\hat{\jmath} + (3 + 3)\hat{k} = \langle -21, -3, 6\rangle$$

Since the plane contains both lines, we can find a point in the plane by finding a point on either line, using any parameter value of t. Taking $t = 0$ in the parametric equations for L_1, we know that $(2, -5, 1)$ is contained in the plane. So an equation for the plane is

$$\langle -21, -3, 6\rangle \cdot \langle x - 2, y + 5, z - 1\rangle = 0$$

or

$$-21x - 3y + 6z + 21 = 0.$$