Math 8
Winter 2015
Power Series

A power series about $x = 1$ is an "infinite polynomial"

$$\sum_{n=0}^{\infty} c_n (x-a)^n.$$

In general, this series will converge for some values of x and diverge for some values of x. The set of values of x for which the series converges is called the *interval of convergence*. (This set always is an interval.)

A power series always has a *radius of convergence* R, with $0 \leq R \leq \infty$. The series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

If $R = \infty$, the series converges for all x, and the interval of convergence is $(-\infty, \infty)$.

If $R = 0$, the series converges only for $x = a$, and the interval of convergence is the "interval" $[a, a]$ containing only one point.

If $0 < R < \infty$, the series may converge or diverge at the endpoints $|x-a| = R$ of the interval of convergence, or it may converge at one endpoint and diverge at the other. The possibilities for the interval of convergence are $[a-R,\, a+R]$, $(a-R,\, a+R]$, $[a-R,\, a+R)$, $(a-R,\, a+R)$.

We can often find the radius of convergence of a power series using the ratio test.

A power series defines a function f whose domain is the interval of convergence of the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n.$$

For example, the function

$$f(x) = \sum_{n=0}^{\infty} (x-a)^n$$

is defined by a geometric series with ratio $r = x - a$. Therefore, it converges when $|x-a| < 1$ (the interval of convergence is $(a-1, a+1)$), and we know what it converges to:

$$f(x) = \sum_{n=0}^{\infty} (x-a)^n = \frac{1}{1-(x-a)}.$$

We can use this power series to find power series expressions for other functions.

For example, suppose we want to express $\dfrac{1}{x+3}$ as a power series about $x = 0$. We know we can write

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Since

$$x + 3 = 3\left(\frac{x}{3} + 1\right) = 3\left(1 - \left(-\frac{x}{3}\right)\right),$$

we can write

$$\frac{1}{x+3} = \frac{1}{3\left(1-\left(-\frac{x}{3}\right)\right)} = \frac{1}{3}\left(\frac{1}{\left(1-\left(-\frac{x}{3}\right)\right)}\right) = \frac{1}{3}\left(\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n.$$

Since we were substituting $-\dfrac{x}{3}$ for x in the power series $\sum\limits_{n=0}^{\infty} x^n$, which converges for $|x| < 1$, we expect convergence for $\left|\dfrac{x}{3}\right| < 1$, or $|x| < 3$.

Exercise 1: Apply the Ratio Test directly to the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^n$$

to find the radius of convergence.

A very important fact about power series is that they can be integrated and differentiated term-by-term, and the resulting power series has the same radius of convergence. (It may behave differently at the endpoints of the interval of convergence.)

For example, taking

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

and integrating both sides, we get

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C.$$

We can plug in $x = 0$ to solve for C:

$$\sum_{n=0}^{\infty} \frac{0^{n+1}}{n+1} = -\ln|1-0| + C.$$

$$0 = 0 + C$$

$$C = 0$$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x|.$$

We know this expression is valid within the radius of convergence of our original series, or for $|x| < 1$.

In particular, it converges at $x = -\dfrac{1}{2}$, and we can write

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^{n+1}}{n+1} = -\ln\left|1-\left(-\frac{1}{2}\right)\right|.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^{n+1}} = -\ln\left|\frac{3}{2}\right| = -(\ln 3 - \ln 2).$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^{n+1}} = \ln 2 - \ln 3.$$

Exercise 2: Apply the error estimate from the Alternating Series Test to the equation

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^{n+1}} = \ln 2 - \ln 3$$

to approximate $\ln 2 - \ln 3$ to within 3 decimal places.

Exercise 3: Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(Note, by convention we define $0!$ to equal 1, so the constant term of this series is 1.)

For the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

use term-by-term differentiation to find a power series expression for $f'(x)$.

You should have found $f'(x) = f(x)$. Did you? What function do you know that has this property?

Exercise 4: Express $\dfrac{1}{x^2+1}$ as a power series about $x = 0$.

Use this expression to express $\tan^{-1}(x)$ as a power series about $x = 0$.

What is the interval of convergence of the resulting power series?

Use the expression for $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ to express $\pi\sqrt{3}$ as an infinite series.

Find a sum of finitely many fractions that approximates $\pi\sqrt{3}$ to within an error of at most .05.