Separable permutations, Robinson-Schensted and shortest containing supersequence

Greta Panova (Harvard)
joint with Andrew Crites (U. Washington) and Greg Warrington (U. Vermont)

Permuation Patterns 2010

- Permutations and Partitions:
  Robinson-Schensted algorithm (RSK) and the Shape of the resulting SYT (Greene's theorem)
- Separable permutations and their shape under RSK
- Supersequences of separable permutations and shapes containment
- Origins: Shortest containing supersequence

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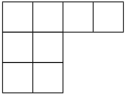
Basics

Partition of n: $\lambda \vdash n$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots), \sum \lambda_i = n$
Young diagram of shape λ:

$$\lambda = (4, 2, 2) \rightarrow$$



Young Tableau of shape λ:

$$\begin{array}{cc} & < \\ \wedge & \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 8 \\ \hline 2 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 5 & 7 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \end{array}$$

Longest Increasing Subsequence of a permutation w: increasing subsequence of W of maximal possible length.

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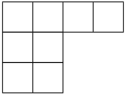

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$w = 78\underline{\mathbf{2}}9\underline{\mathbf{35}}41\underline{\mathbf{6}}$, 2356

RSK definition

Bijection: $w \rightarrow (\underbrace{P}_{\text{Insertion Tableau}}, \underbrace{Q}_{\text{Recording Tableau}})$.

$w_1 \ldots w_i \rightarrow (P_i, Q_i)$

$P_{i+1} = w_{i+1} \rightarrow P_i$: $Q_{i+1} = Q_i + \boxed{\text{i+1}}$@new box of P_{i+1}

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$w = 561423$

$561\underline{\mathbf{4}}$, $\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 5 & 6 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\right)$

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Shape of the Tableaux under RSK

Theorem (Greene)

Let $w \in S_n$, $(P, Q) = \mathrm{RSK}(w)$ *and* $\lambda = \mathrm{sh}(P) = \mathrm{sh}(Q)$. *We have that for every* i

$$\lambda_1 + \cdots + \lambda_i = |v^{i1}| + \cdots + |v^{ii}|,$$

where $v^{i1}, \ldots, v^{ii}$ *are* i *disjoint increasing subsequences of* w *of maximal total length.*

Example

$w = 236145$

$i = 1$: $v^{11} = 2345 \rightarrow \lambda_1 = 4$

$i = 2$: $v^{12} = 236, v^{22} = 145 \longrightarrow \lambda_1 + \lambda_2 = 3 + 3 \rightarrow \lambda_2 = 2$

$\lambda = (4, 2)$ Indeed, $\mathrm{RSK}(236145) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \right)$

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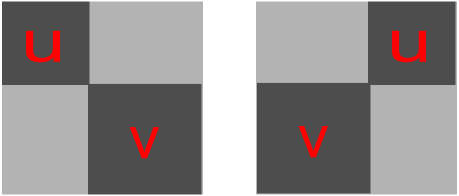

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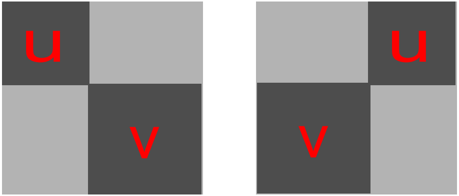

Example

Separable: 65478213,
Not separable **2**3**614**5 : 2614 $\sim$ 2413

Main Result

Theorem

If σ is a separable permutation and $\lambda = \mathrm{sh}(\sigma)$, there are disjoint increasing subsequences $v^1, v^2, \ldots$, s.t. $\bigcup v^i = \sigma$ and $\lambda_i = |v^i|$ for all i.

In view of Greene's theorem where

$$\lambda_1 + \cdots + \lambda_i = |v^{i1}| + \cdots + |v^{ii}|,$$

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Example

$\sigma = 541237698$
$v^{11} = 12378 : 54\underline{\mathbf{1237}}69\underline{\mathbf{8}} \rightarrow \lambda_1 = 5$
$v^{21} = 12378, v^{22} = 469 :\rightarrow \lambda_2 = 3$
$v^{31} = 12378, v^{32} = 469, v^{33} = 5 :\rightarrow \lambda_3 = 1$

Proof setup: the Inversion Poset

Definition

The **inversion poset** of a permutation w is the poset on elements (i, w_i) under the partial order:

$$(a, b) \preceq (c, d) \Leftrightarrow (a \leq c) \wedge (b \leq d).$$

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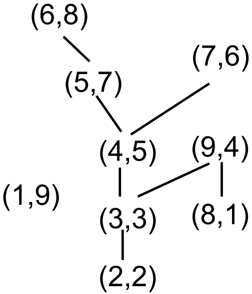
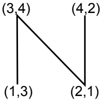


The Inversion Poset of a separable permutation

3142 ⟶


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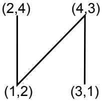

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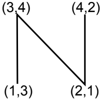
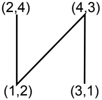
3142 ⟶ 

2413 ⟶ 

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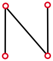
The inversion poset of a separable permutation is $N-$free, i.e. it does not contain a subposet isomorphic to .

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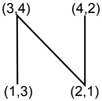
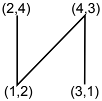
3142 ⟶ 

2413 ⟶ 

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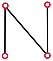
The inversion poset of a separable permutation is N−free, i.e. it does not contain a subposet isomorphic to .

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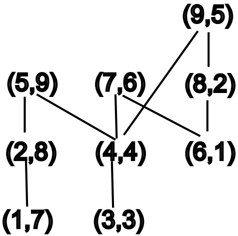

The Inversion Poset of a separable permutation

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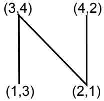

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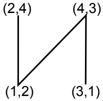

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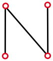
The inversion poset of a separable permutation is $N-$free, i.e. it does not contain a subposet isomorphic to .

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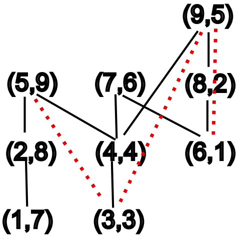

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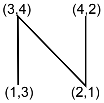

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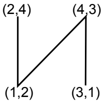

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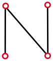
The inversion poset of a separable permutation is N−free, i.e. it does not contain a subposet isomorphic to .

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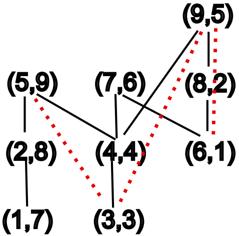

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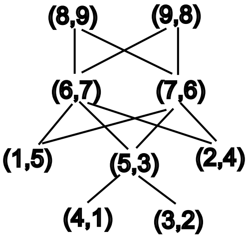

Proving the Main Theorem

Lemma

Let u, v, and w be increasing subsequences of a separable permutation. If $w \bigcap v = \emptyset$ and both intersect u, there exist two disjoint subsequences α and β, such that

- $\alpha \bigcup \beta = w \bigcup v$,
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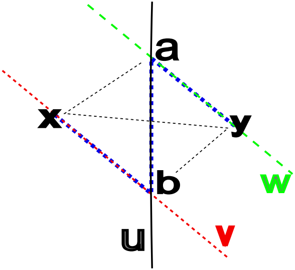

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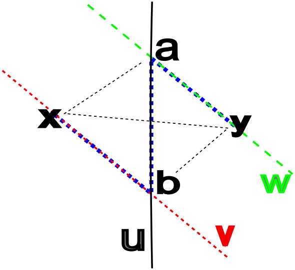
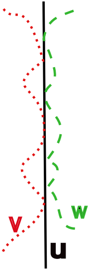


General picture.



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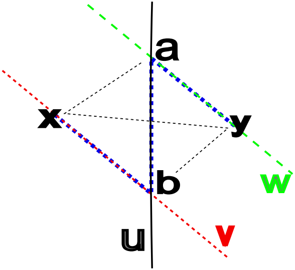

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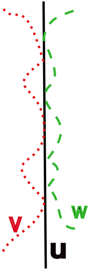

Needs formalization.

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Argue by contradiction.

Let $C_1 = u \cap (v \cup w)$ and assume there is no chain $\beta \subset (v \cup w)$, s.t. $C_1 \subset \beta$ and $(v \cup w) \setminus \beta$ is also a chain.

Let $C \subset (v \cup w)$ be some maximal chain, s.t. $C_1 \subset C$. Then there exist $x, y \in (v \cup w) \setminus C$, s.t. $x \ngtrless y$. Then $x \in v$, $y \in w$.

By maximality, $x \cup C$, $y \cup C$ are not chains, so there are $a, b \in C$, s.t. $x \ngtrless a$, $y \ngtrless b$, so $a \in w$ and $b \in v$. Assume $a \succ b$, then we must have $x \succ b$ and $y \prec a$.

We have

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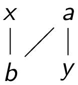

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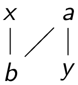

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Towards the Main Theorem

Proposition

Let $u = \{u^1, \ldots, u^k\}$ *be* k *disjoint increasing subsequences (***d.i.s.***) of a separable permutation* σ *with* $\mathrm{sh}(\sigma) = \lambda$. *Then there is an increasing subsequence* w^{k+1}, *disjoint from* u^i*s, s.t.* $|w^{k+1}| \geq \lambda_{k+1}$.

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Let V — $k+1$ d.i.s., $\sum_{v \in V} |v| = \sum_{i=1}^{k+1} \lambda_i$ by Greene's thm.

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$$\tilde{V}_1 = \{\tilde{v}^i : \tilde{v}^i \bigcap u^1 = \emptyset, i > 1; \bigcup \tilde{v}^i = \bigcup_{v \in V_1} v\}.$$

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Repeat with $U \setminus \{u^1\}$ and $(V \setminus V_1) \bigcup \tilde{V}_1 \setminus \{\tilde{v}^1 | v^1 \bigcap u^1 \neq \emptyset\}$.

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End up with

$$\tilde{V} = \{w^i | w^{k+1} \bigcap u^j = \emptyset, 1 \leq j \leq k\}$$

and

$$\bigcup_i w^i = \bigcup_{v \in V} v.$$

$$|w^{k+1}| = \lambda_1 + \dots + \lambda_{k+1} - \underbrace{|w^1| - \dots - |w^k|}_{\leq \lambda_1 + \dots + \lambda_k, \text{ Greene's thm}} \geq \lambda_{k+1}$$

□

Proof of the Main Theorem

In Proposition, let $|u^1| + \cdots + |u^k| = \lambda_1 + \cdots + \lambda_k$ by Greene's thm.
Then $u^1, \ldots, u^k, w^{k+1}$ — $k + 1$ d.i.s., total length $\geq \lambda_1 + \ldots + \lambda_{k+1}$, so $|w^{k+1}| = \lambda_{k+1}$.
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Super- and subsequences and shapes containment

Question

If σ is a subsequence of w, when is $\mathrm{sh}(\sigma) \subset \mathrm{sh}(w)$*?*

Example

$w = \underline{\mathbf{24}}2\underline{\mathbf{13}}$

$$\left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \right)$$

$\sigma = 2413$

$$\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} , \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \right)$$

$$\mathrm{sh}(\sigma) = (2,2) \not\subset \mathrm{sh}(w) = (3,1,1)$$

Separable subsequence

Corollary (to Main Theorem)

If a word w contains a separable permutation σ as a pattern, then $\mathrm{sh}(\sigma) = \lambda \subset \mathrm{sh}(w) = \mu$.

Proof.

Greene's theorem: $|w^1| + \cdots + |w^k| = \mu_1 + \cdots + \mu_k$. Set $u^i = w^i \bigcap \sigma$.
By proposition there exists u^{k+1} i.s. in σ, $|u^{k+1}| \geq \lambda_{k+1}$.
In w: $w^1, \ldots, w^k, u^{k+1}$ — d.i.s, so

$$\underbrace{|w^1| + \cdots + |w^k|}_{\mu_1 + \cdots + \mu_k} + \underbrace{|u^{k+1}|}_{\lambda_{k+1}} \leq \mu_1 + \cdots + \mu_{k+1}$$

$$\Longrightarrow \lambda_{k+1} \leq \mu_{k+1}$$

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Supersequences

A word w is a **supersequence** of σ if it contains σ as a subsequence.

Definition

Let $B \subset S_n$, w is a supersequence of B if w is a supersequence of each element of B. Let $\mathrm{SCS}_n(B)$ be the minimal length of a supersequence of B.

Question

Bounds for $\mathrm{SCS}_n(S_n)$? Bounds for $\mathrm{SCS}_n(B)$ for certain sets B?

Koutas, Hu, 1975: $\mathrm{SCS}_n(S_n) \leq n^2 - 2n + 4$,
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Supersequences of Separable Permutations

$B = \{\sigma_1, \dots, \sigma_k\}$, σ_i — separable; w — supersqn(B).

$$\bigcup_i \mathrm{sh}(\sigma_i) \subset \mathrm{sh}(w) \Longrightarrow \mathrm{SCS}_n(B) \geq |\bigcup_i \mathrm{sh}(\sigma_i)|$$

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Take $rw(T)$, T - superstandard, $\mathrm{sh}(T) = \lambda$ Example: $\lambda = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; & \; & \; \\ \cline{1-3} \; & \; \\ \cline{1-2} \end{array}$,

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2} \end{array}, \mathrm{rw}(T) = 895671234$$

Note: $\mathrm{rw}(T)$- separable (in fact 213-avoiding), $\mathrm{sh}(\mathrm{rw}(T)) = \mathrm{sh}(T)$.

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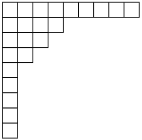
Take $B = \{\mathrm{rw}(T) | \mathrm{sh}(T) = \lambda, \lambda \vdash n\}$ $\mu(n) = \bigcup_{\lambda \vdash n} \lambda$

Example

$n = 9$, $k = 5$. $B = \{\sigma_1, \ldots, \sigma_5\}$:

$\mathrm{sh}(\sigma_1 = 123456789) = (9)$,
$\mathrm{sh}(\sigma_2 = 678912345) = (5, 4)$,
$\mathrm{sh}(\sigma_3 = 789456123) = (3, 3, 3)$,
$\mathrm{sh}(\sigma_4 = 978563412) = (2, 2, 2, 2, 1)$,
$\mathrm{sh}(\sigma_5 = 987654321) = (1, 1, 1, 1, 1, 1, 1, 1, 1)$.



; $|\bigcup_{i=1}^{5} \mathrm{sh}(\sigma_i)| = 23$.

w=6 9 7 8 7 5 9 6 5 4 3 1 2 3 4 5 6 7 8 9 1 2 3,

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Proposition

$|\mu(n)| \sim n(\ln n + \cdots)$ and number of corners is $\tilde{2}\sqrt{n}$, so B has $\tilde{2}\sqrt{n}$ permutations and $\mathrm{SCS}_n(B) \geq n(\ln n + \cdots)$

Related Questions, work in progress

Question

If σ is a subword of w, characterize when $\mathrm{sh}(\sigma) \subset \mathrm{sh}(w)$.

Conjecture

If $\sigma = uv$, where $u = (w_i | i < k, w_i < k)$, $v = (w_i | i \geq k, w_i \geq k)$, then
$$\mathrm{sh}(\sigma) \underbrace{=}_{obvious} \mathrm{sh}(u) + \mathrm{sh}(v) \underbrace{\subset}_{?} \mathrm{sh}(w).$$

Question

For what shapes λ is Greene's theorem "satisfied" in the sense of: For every permutation w, $\mathrm{sh}(w) = \lambda$, there exist d.i.s. v^i, s.t. $|v^i| = \lambda_i$.

T	*h*	*a*	*n*	*k*
y	*o*	*u*		
!				