Recurrence vs Transience in Dimensions 2 and 3

Lin Zhao
Department of Mathematics
Dartmouth College
March 2011

In 1921 George Polya investigated random walks on lattices. If upon reaching any vertex of the lattice, the probability of choosing any one of the $2d$ edges leading out of that vertex is $1/2d$, then this walk is called simple random walk in d dimensions. The question that Polya posed amounts to this: Is the traveling point certain to return to its starting point during the course of its wanderings? If so, we say that the walk is recurrent. If we denote the probability that the point never returns to its starting point by p_{escape}, then the chain is recurrent if and only if $p_{escape} = 0$. If not, we say that the walk is transient. In other words there is a positive probability that the point will never return to its starting point. The chain is transient if and only if $p_{escape} > 0$.

Polya proved the following theorem:

POLYA's Theorem. Simple random walk on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$.

We will try to understand this theorem by exploiting the connections between questions about a random walk on a graph and questions about electric currents in a corresponding network of resistors. This approach, by calling on methods that appeal to our physical intuition, will leave us feeling that we know "why"the theorem is true.

We can determine the type of an infinite lattice from properties of bigger and bigger finite graphs that sit inside it. The lattice analog of balls (solid spheres) in space is defined as follows: Let r be an integer, which will be the radius of the ball. Let $G^{(r)}$ be the graph gotten from Z^d by throwing out vertices whose distance from the origin is $> r$. By "distance from the origin" we mean the length of the shortest path along the edges of the lattice between the two points. We adopt this defination of distance because if an electron wants to travel from the origin to certain node, the path it makes is not a line segment conneting the origin and the node, which gives to the usual Euclidean distance. Instead, it consists of several edges and the distance we defined above is the smallest number of edges which all together connect the origin and the desired point the electron aims at. Let $S^{(r)}$ be the "sphere" of radius r about the origin, i.e. those points that are exactly r units from the origin.

In $d = 2$, $S^{(r)}$ looks like a rotated square. If we set up a $x - y$ coordinates system in the plane with $(0, 0)$ point located at the origin $(\mathbf{0})$ and make x and y axises parallel to some edges, the equation for $S^{(r)}$ is

$$|x| + |y| = r \tag{1}$$

Similarly, in the case of $d = 3$, $S^{(r)}$ looks like an octahedron and its equation is

$$|x| + |y| + |z| = r \tag{2}$$

Electrical Formulation of the Type Problem: To determine $p^{(r)}_{escape}$ electrically, we simply ground all the points of $S^{(r)}$, which is the boundary of $G^{(r)}$, maintain $\mathbf{0}$ at one volt, and measure the current $i^{(r)}$ flowing into the circuit.We have, from section 3.4 of referrence 1,

$$p^{(r)}_{escape} = \frac{i^{(r)}}{2d} \tag{3}$$

Since the voltage being applied is 1, $i^{(r)}$ is just the effective conductance between **0** and $S^{(r)}$, i.e.,

$$i^{(r)} = \frac{1}{R^r_{EFF}} \tag{4}$$

where $R^{(r)}_{EFF}$ is the effective resistance from **0** to $S^{(r)}$.
Define R_{EFF}, the effective resistance from zero to infinity to be the limit of $R^{(r)}_{EFF}$. Then

$$p_{escape} = \frac{1}{2dR_{EFF}} \tag{5}$$

Thus the walk is recurrent if and only if the resistance to infinity is infinity.

One Dimension is Easy: Since an infinite line of 1-ohm resistors obviously has infinite resistance, it follows that simple random walk on the 1-dimensional lattice is recurrent, as stated by Polya's theorem.

What about Higher Dimensions? The difficulty is that the d-dimensional lattice Z^d with the distance we defined lacks the rotational symmetry. By solving the appropriate discrete Dirichlet problem, the voltages for a one-volt battery attached between **0** and the points of $S^{(3)}$ in Z^2. The resulting voltages are:

$$\begin{array}{c} 0 \\ 0\ .091\ 0 \\ 0\ .182\ .364\ .182\ 0 \\ 0\ .091\ .364\ 1.00\ .364\ .091\ 0 \\ 0\ .182\ .364\ .182\ 0 \\ 0\ .091\ 0 \\ 0 \end{array}$$

The voltages at points of $S^{(1)}$ are equal, but the voltages at the points of $S^{(2)}$ are not. This means that the resistance from **0** to S^3 cannot be simply written as the sum of the resistances from **0** to $S^{(1)}$,$S^{(1)}$ to $S^{(2)}$, and $S^{(2)}$ to $S^{(3)}$.

In order to get around the lack of rotational symmetry of the lattice, we use Rayleigh's method which involves modifying the network whose resistance we are interested in so as to get a simpler network. We consider two kinds of modifications, shorting and cutting. Cutting involves nothing more than clipping some of the branches of the network, or what is the same, simply deleting them from the network. Shorting involves connecting a given set of nodes togher with perfectly conducting wires, so that current can pass freely between them. In the resulting network, the nodes that were shorted together behave as if they were a single node.

The usefullness of these two procedures stems from the following observations:

Shorting Law: Shorting certain sets of nodes together can only decrease the effective resistance of the network between two given nodes.

Cutting Law: Cutting certain branches can only increase the effective resistance between tow given nodes.

Rayleigh's idea was to use the Shorting Law and Cutting Law above to get lower and upper bounds for the resistance of a network. In the case $d = 2$, we will modify the 2-dimensional resistor network by shorting certain sets of nodes together so as to get a new network whose resistance is readily seen to be infinite. As shorting can only decrease the effective resistance of the network, the resistance of the original network must also be infinite. Thus the walk is rcurrent when $d = 2$.

In the case $d = 3$, we will modify the 3-dimensional network by cutting cut certain of the resistors so as to get a new network whose resistance is readily seen to be finite. As cutting can only increase the resistance of the network, the resistance of the original network must also be finite. Thus the walk is transient when $d = 3$.

The Plane is Easy: When $d = 2$, we short together nodes on squares about the origins, that is, for any given radius r we connect all the points on $max\{|x|, |y|\} = r$ and view them as node r. The network we obtain is equicalent to a network on a line with $8n + 4$ edges between node n and $n + 1$. Now as n 1-ohm resistors in parallel are equivalent to a single resistor of resistance $1/n$ ohms, the resistance of theis network out to infinity is

$$\sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty \tag{6}$$

As the resistance of the old network can only be bigger, we conclude that it too must be infinite, so that the walk is recurrent when $d = 2$. Actually, by the symmetry of the original network, points on $max\{|x|, |y|\} = r$ for a given r are at the same potential, so the effective resistance is not affected by shorting these points together.

When $d = 3$, we want to searching for a residual network. What we want to do is to delete certain of the branches of the network so as to leave behind a residual network having manifestly finite resistance. The problem is to reconcile the "manifestly" with the "finite". We want to cut out enough edges so that the effective resistance of what is left is easy to calculate, while leaving behind enough edges so taht the result of the calculation is finite.

Trees are easy to analyze: Trees-that is, graphs without circuits- are the easiest to work with. A full binary tree is one splits into two edges at the root and nodes of every generation. By shorting nodes in the same generation together, we can show that full binary has finite resistance. Unfortunately, we can't even come close to finding the full binary trees as a subgraph of the 3-dimensional lattice. For in this tree, the number of nodes in a "ball" of radius r, that is, all the n^{th} generations with $n \le r$, grows exponentially with r, whereas in a d-dimensional lattice, it grows like r^d, i.e., much slower. There is simply no room for the full binary trees in any finite-dimensional lattice.

***NT_3*, a "Three-Dimensional" tree**: A 3-dimensional tree NT_3 is one where the number of nodes within a radius r of the root is on the order of r^3. Let's look at an easier tree first, NT_2, a 2-dimensional tree. The idea behind NT_2 is that, since a ball of radius r in the plane, ought to contain something like r^2 points, a sphere of radius r ought to contain something like r points, so the number of points in a sphere should roughly double when the radius of the sphere is doubled. For this reason we make the branches of our tree split in two every time the distance from the origin is (roughly) doubled. Mathematically, the nodes in the r^{th} generation split in two when $r = 2^n - 1$ for some positive integer n. Thus r is the radius of the ball, and $S^{(r)}$, which in this case is the r^{th} generation, contains 2^r nodes.

NT_3, a "Three-Dimensional" tree: Similarly, in a 3-dimensional tree, when we double the radius, the size of a sphere should roughly quadruple. Thus in NT_3, we make the branches of our tree split in four where the branches of NT_2 would have split in two. That is, the nodes in the r^{th} generation split in fout when $r = 2^n - 1$ for some positive integer n. Thus $S^{(r)}$, the r^{th} generation, contains 4^r nodes.

By the same shorting technique used in full binary tree, we get that the resistance of NT_2 is infinite, but the resistance of NT_3 is not, as we would hope.

But does NT_3 Fit in the Three-Dimensional Lattice? We would like to embed NT_3 in Z^3. We start by trying to embed NT_2 in Z^2. To contruct this picture, we start from the origin and draw a rays, one going north, one going east. Whenever a ray intersects the line $x + y = 2^n - 1$ for some n which is $S^{(r)}$ in the plane, it spilts into 2 rays, one going north, and one going east.

Identifications: This isn't really an embedding, since certain pairs of points that were distinct in NT_2 get identified, that is, they are made to correspond to a single point in the picture. In terms of our description, sometimes a ray going north and a ray going ease pass through each other. However, because the points of each identified pair are at the same distance from the root of NT_2, when we put a battery between the root and the n-generation, they will be at the same potential. Hence, the current flow is not affected by these identifications, so the identifications have no effect on R_{EFF}. For our purpose, then, we have done just as well as if we had actually embedded NT_2.

Now looking at NT_2, it is clear that all node at the same $S^{(r)}$ for a given r are at the same potential. So by embeding NT_2 into $Z^{(2)}$, we choose a substantial amount of nodes with the same potential on each $S^{(r)}$ and get rid of other nodes which have different potentials on that sphere.

Not NT_3 **?**: To construct the analogous picture in three dimensions, we start three rays off from the origin going north, east, and up. Whenever a ray intersects the plane $x + y + z = 2^n - 1$ for some n which is $S^{(r)}$ in the space, it splits into three rays, going north, east, and up. But the subgraph of the 3-dimensional lattice obtained in this way is not NT_3. Because it splits into three instead of four.

We call this tree $NT_{2.5849...}$ because it is 2.5849...-dimensional in the sense that when you double the radius of a ball, the number of points on the sphere gets multiplied 3 and

$$3 = 2^{log_2 3} = 2^{2.5849...-1} \tag{7}$$

Again, certain pairs of points of $NT_{2.5849...}$ have been allowed to correspond to the same point in the lattice, but once again the intersections have no effect on R_{EFF}.

$NT_{2.5849...}$: So we haven't come up with our embedded NT_3 yet. But, the resistance of

$NT_{2.5849\ldots}$ out to infinity is

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 \tag{8}$$

Thus we have found an infinite subgraph of the 3-dimensional lattice having finite resistance out to infinity, and we are done.

Referrences

1. Peter G. Doyle and J. Laurie Snell (1984), Random Walks and Electric Networks. Chapter 3, 5-6.