Random-Turn Hex and Other Selection Games

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Abstract

In this report, we summarize the results in [6]. We define a mathematical formulation for all selection games. When considering the random-turn selection game, we prove that the result of the game equals to two players play randomly. By using the theory of Boolean functions, we show that the average number of turns before determining the winner of Random-Turn Hex is at least $L^{3/2+o(1)}$, where L is the size of the board.

1 Introduction

The **Hex** game is invented independently by Piet Hein in 1942 and John Nash in 1948. This game is a two players game and is played on a hexagonal grid. Two players alternatively chose one hexagon on the board. A player wins when connecting the two opposite sides.

Although we can prove that the first player has a winning strategy [1], finding an optimal strategy is not easy. In general, determining whether player has a winning strategy is PSPACE-complete [8]. Arneson et al. solved the Hex on board of size up to 9×9 [4]. However, the Hex game is usually played on an 11×11 board and there is no known winning strategy.

The rule of Random-Turn Hex is the same as ordinary Hex. Nevertheless, instead of moving alternatively, two players decide who can take action by toss a coin. Peres et al. analyzed the optimal strategy of Random-Turn Hex and generalized the analysis to other selection games [6]. This paper proved that any optimal strategy for one player is also an optimal strategy for the other player in any random-turn selection game.

We will introduce the background knowledge in the section 2. In the section 3, we will find out the property of the optimal strategy for random-turn selection game and how to find the optimal strategy. The expected number of turns before determining the winner is analyzed in the section 4. Finally, we will summarize the result in the last section.

2 Preliminary

In this section, we will build a model for the selection game. We can formalized a class of games as follows. Let S be a set of a size n. Define f as a function from a subset of S to a real number. The game is played between two players. The first player picks one of the n elements. Then the second player choose one of the remaining $n-1$ elements, and so on. This game ends when all the elements are chosen. Let the sets S_1 and S_2 be the sets selected by these two players, respectively. The first player earn $f(S_1)$ and the other player receives $-f(S_1)$. We call this type game as **selection game**.

Obviously, Hex is one example of the selection game. The set S is the elements on a $L \times L$ hexagonal grid. The function $f(S_1)$ is 1 when S_1 connects left to right. Otherwise, $f(S_1)$ is -1.

After formalizing the selection game, we want to find the optimal strategy for a random-turn selection game. We say M is a strategy for a random-turn selection game, if M is a function from two disjoint subsets (T_1, T_2) of S to elements of S. Intuitively, M denoting the elements that player would choose when the first player picked the elements in T_1 and the second player picked the elements in T_2. We say a strategy M is optimal if M can maximize the expected payoff for the player.

Assume that two players play optimally. We can represent the expected payoff of the first player in one turn by $E(T_1, T_2)$. We can compute E inductively.

3 Optimal Strategy

In this section, we will find out the basic property of the optimal strategy for random-turn the selection game. Suppose that one of the player has an optimal strategy, then the other player also can apply this strategy. This is due to two players have equal probability to take action in each turn. Consequently, when two players play the same strategy, each element is picked by one of the player with probability 1/2. Thus, we can prove

Theorem 3.1. *[6] The value $E(\emptyset, \emptyset)$ of a random-turn selection game is the expectation of $f(T)$, where $T \subset S$ is selected uniform randomly.*

After proving the value of of the random-turn selection game, the next reasonable question is how to find a optimal strategy for Random-Turn Hex. Although we know the value, finding an optimal strategy is still an open problem. However, there is a way to find an approximate optimal strategy for one special class of random-turn selection game.

We say a selection game is a **win-or-lose game**, if $f(T)$ is either one or negative one. Moreover, we say a game is a **monotone game**, if f is monotone. That means $f(S_1) \geq f(S_2)$, where $S_2 \subseteq S_1$. We define an element s as **pivotal element** for $S_1 \subset S$, if $f(S_1 \cup \{s\}) \neq f(S_1 \setminus \{s\})$. Intuitively, a pivotal element s for S_1 is more important than any other elements, because it might effect the value of f.

Apparently, Hex is a monotone, win-or-lose game. For any monotone, win-or-lose, random-turn selection game, we can prove the optimal strategy has the property as follows:

Theorem 3.2. *[6] Let M be an optimal strategy for a monotone, win-or-lose, random-turn selection game. When the sets T_1 and T_2 are selected by two players respectively, $M(T_1, T_2) = s$, if and only if s is most likely to be a pivotal for $T_1 \cup T$, where $T \subseteq S \setminus (T_1 \cup T_2)$ is selected uniform at random.*

Although we cannot find an optimal strategy, we can use this theorem to find an approximate good strategy. When two players have already selected T_1 and T_2 respectively, we can randomly sample several subsets of $S \setminus (T_1 \cup T_2)$ and estimate which element is most likely to be a pivotal. It can be shown that we can found a move that is within $O(\epsilon/L^2)$ of being optimal, except with probability $O(\epsilon/L^2)$, where ϵ is fixed and L is the size of the board [6]. Additionally, one of the author developed the software Hexamania, which can be downloaded on the website [2]. We can play the Random-Turn Hex game with computer, whose strategy is based on this method.

4 Number of turns for win-or-lose games

For win-or-lose games, the game usually ends when one of the player wins. How many turns before the winner is determined in expectation when both players play optimally? We will give a lower bound of the expected number of the turns in this section.

We can consider the selection game as evaluating the value of the function f. Originally, the function f is a function from a subset of S to a real number. Nonetheless, we can use different representation for a subset of S. Let $\vec{x}$ be a vector of a size $|S| = n$. Each element x_i in the $\vec{x}$ has value either 1 or -1, denoting whether the ith element in S is belong to the first player or the second player. Hence, determining the winner equals computing the value of $f(\vec{x})$. In this way, the number of turns is equivalent to the number of elements in $\vec{x}$ we should examined before $f(\vec{x})$ is computed.

Let $I_i(f)$ be the probability that change the value of x_i will change the value of $f(\vec{x})$. Combing the result from O'Donnell and Servedio [5], we can get the following inequality

Theorem 4.1. *[6] For any win-or-lose, random-turn selection game, when two players play optimally, the expected number of turns before determining the winner satisfies the following inequality*

$$E[\textit{number of turns}] \geq (\sum_i I_i(f))^2$$

For the Random-Turn Hex game, the probability $I_i(f)$ is related to the percolation (filling in the empty hexagons randomly). Hence, we can apply the theory of percolation and get $I_i(f)$ is $L^{-5/4+o(1)}$ [9], where L is the size of the board. Consequently, we can get the following upper bound for the Random-Turn Hex game.

Corollary 4.2. *[6] For the Random-Turn Hex, when two players play optimally on a $L \times L$ board, the expected number of turns before determining the winner is at least $L^{3/2+o(1)}$.*

Since the usual board size of Hex is 11×11, we know the expected number of turn is at least 36.

5 Summary

In this report, we gave a model for the selection games. Moreover, we proved that the value of the random-turn selection game is equivalent to two players play randomly. For the win-or-lose, monotone, random-turn selection game, we identified the property of the optimal strategy. Moreover, combining the theory of Boolean functions and percolation, we can get the lower bound of the expected number of turns. Specifically, for the Random-Turn Hex game, the expected number of turns is at least $L^{3/2+o(1)}$.

Although the game GO is not a selection game, because one player can remove the other player's pieces, there is a board game, which is played on the board of GO, which is called

Gomoku or Five in a Row. The rules of Gomoku are simple. Two players alternatively make moves and the first player who constructs a line with five pieces wins. In general, determining whether one player has a winning strategy is a PSPACE-complete problem [7]. However, we can model Gomoku as a selection game. Let S be the sites on the 19×19 board. The payoff function $f(S_1, S_2)$ is the number of lines with length five in S_1 minus the number of lines with length five in S_2. When considering the Random-Turn Gomoku game, we can apply the theorem 3.1 and get the value is equal to two players play randomly. However, in this model, although Gomoku is monotone, it is not a win-or-less selection game. Hence, we cannot apply the theorem 3.2 to find the optimal strategy.

As we can see, Gomoku is a generalized version of Full-board tic-tac-toe game mentioned in [6]. Since the random-turn selection is related to the percolation, maybe we can change the definition of the pivotal element. Let's call a element s pivotal for $S_1 \subset S$, if $E(S_1 \cup \{s\}, S_2)$ is maximized over all the elements in $S \setminus (S_1 \cup S_2)$. Perhaps, we can create the connection between the pivot element and the optimal strategy.

Furthermore, there is another game, which is played on the board of GO, called Connect6 [3]. The rule of Connect6 is similar to Gomoku and the first player who connected a line of a length 6 wins. However, Connect6 has a different rule. Each player will alternatively take two moves. Contrary to the Hex and Gomoku, there is no proof that determining a player has a winning strategy is PSPACE-complete. Moreover, Connect6 seems be fair for two players, in the sense that each player has one more piece after each turn. Although Connect6 is not a selection game, we still can generalize the model of the selection game to include the game Connect6. It also might be interesting to analyze optimal strategy for Random-Turn Connect6.

References

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