Electric Networks and Commute Time

Zachary Hamaker

March 17, 2011

Abstract

The equivalence of random walks on weighted graphs with reversible Markov chains has long been known. Another such correspondence exists between electric networks and these random walks. Results in the language of random walks have analogues in the language of electric networks and visa versa. We outline this correspondence and describe how to translate several of the major terms of electric networks into the language of random walks.

Given a random walk on the weighted graph $G = (V, E)$ and $u, v \in V$, the commute time C_{uv} between u and v is the expected number of steps for a random walk starting at u to pass through v and return to u. The cover time C_G is the expected number of steps for a random walk to hit every point. In the second section, we prove that C_{uv} is equal to the product of $2|E|$ and the effective resistance between u and v in the corresponding electric network following an approach first demonstrated in [1]. The proof highlights several ideas previously discussed. We then use this result to prove bounds for C_G.

1 Random Walks and Electric Networks

The existence of a correspondence between random walks on graphs and electrical networks was first established in [4]. Both systems are governed by graphs which have values attached to the edges, weight and resistance respectively. More importantly, the physical process electric networks model is intimately related to walks on graphs. An electrical network governs the aggregate flow of electrons through the circuit. However, the behavior of an individual electron is thought to approximate a random walk through the nodes of the network. We can then view the electric network as Monte Carlo approximation of this random walk on a scale that eliminates all detectable error. With this in mind, we set out to discover how the aggregate behavior defines the underlying random walk.

Our initial objective is to relate the weights on edges of the graph with the resistance of edges in the electric network. Let $G = (V, E)$ be a connected weighted graph. Here, each

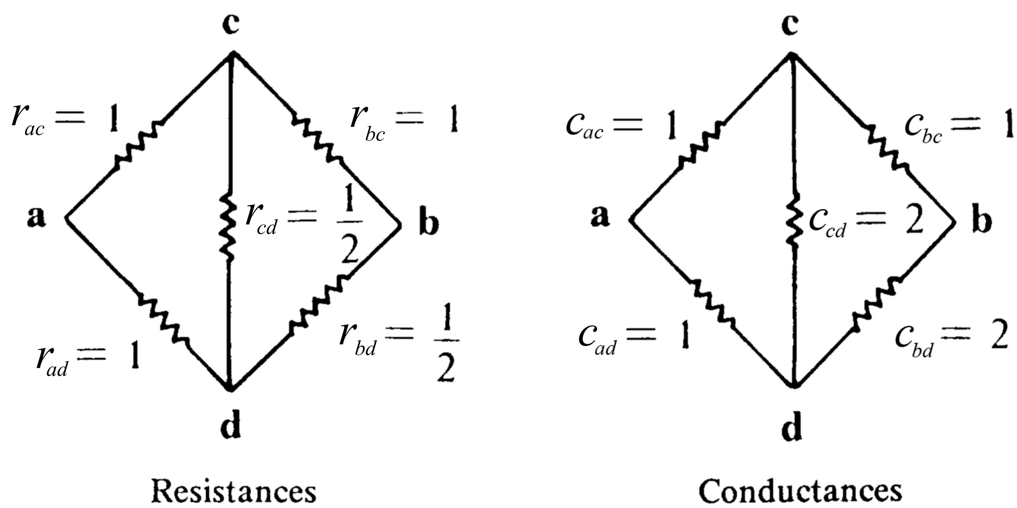

Figure 1.

edge is represented by the pair (e_{uv}, c_{uv}) where $u, v \in V$ and $c_{uv} > 0$ is the weight of the edge between them. Let $N(u)$ denote the neighbors of u in G. Recall the random walk on G is the Markov chain with state space V and transition probability

$$P_{uv} = \frac{c_{uv}}{\sum_{w \in N(u)} c_{uw}}. \tag{1}$$

Our choice to denote the weight of an edge with c is not incidental. Let $\mathcal{N}$ be an electric network with underlying graph G. For $u, v \in V$ with $e_{uv} \in E$, we define r_{uv} to be the resistance on e_{uv}. Then the *conductance* of e_{uv} is $c_{uv} = 1/r_{uv}$. As defined, the capacitance corresponds directly to the weight of e_{uv} in G. Note that as weight encourages passage across an edge while resistance reduces the chance an edge is traversed, this choice makes an intuitive sense which will be born out in later computation.

We work through an example presented in [2], from which Figures 1 and 2 are appropriated. In Figure 1, we see an electric network and the corresponding weighted graph. Figure 2 shows the resulting probabilities for transition along each edge computed using (1). Note that from the transition probabilities, we can construct all of the conductances up to a constant factor.

In this construction, the probabilistic significance of resistance is readily understood. However, in order to discuss voltage or current we need to introduce electricity to our network. The choice of how to do so will effect what these quantities measure. Perhaps the simplest way to do so would be to put a one volt battery connecting some vertices a and b so that the respective voltages $v_a = 1$ and $v_v = 0$. In this case, the voltage at some $u \in V$ would indicate

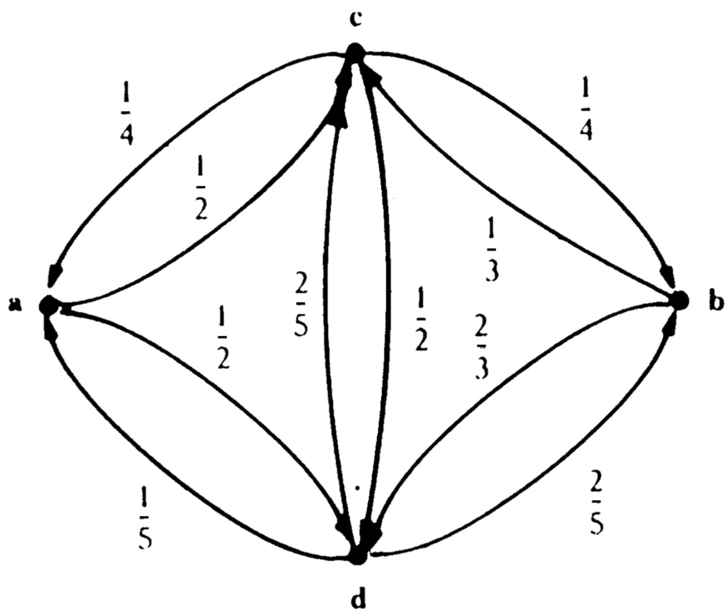

Figure 2.

the probability of a random walk starting at u of hitting a before b. In general, voltages correspond to some sort of hitting time.

Viewing an electric network as the aggregate behavior of electrons, one might suspect that current corresponds in some way to how often an edge is traversed. Such a suspicion is correct. In particular, when a unit current flows into some a and out of some b, the current i_{uv} across the edge e_{uv} corresponds to the expected net number of times that edge is traversed in a random walk starting at a and terminating upon arrival at b. Note, we then have $i_{uv} = -i_{vu}$ as one would expect.

These results are proved by establishing how both interpretations depend on their neighbors and reducing these dependences to two systems of linear equations, each of which has the same unique solution. We will see an example of this proof technique in Theorem 2.1.

2 Commute Time as Resistance

Given a graph $G = (V, E)$ with $u, v \in V$ recall the *hitting time* H_{uv} is the expected number of steps for a random walk on G starting at u to first enter v. We define the *commute time* to be $C_{uv} = H_{uv} + H_{vu}$, that is the expected number of steps for the random walk on G to go from u to v and back again. Before we can determine how to compute the commute time, we must discuss some basic properties of electric networks.

Several important laws govern the relationship between current, voltage and resistance. Perhaps the most fundamental is Ohm's law, which states that voltage is current times resistance, i.e. $V = IR$. We will also need Kirchoff's law, which states $\sum_{v\in N(u)} i_{uv} = 0$ for all $u \in V$. This can be loosely interpreted as saying what goes in must come out; an electron cannot stay at u. Finally, the effective resistance R_{uv} between vertices u and v, not necessarily connected, can be computed using the rule that circuits in series are additive in resistance while circuits in parallel are additive in conductance. This resistance describes the overall difficulty of getting from u to v or visa versa.

As an example of how effective resistance is computed, we find R_{ac} in Figure 1. First, we combine the edges e_{db} and e_{bc} into a new edge with resistance $3/2$ between d and c. There are now two edges between these vertices with conductances 2 and $2/3$ respectively, so combining them we have a single edge e'_{cd} with total conductance of $8/3$, hence resistance $3/8$. This edge is combined with e_{ad} to form a new edge between a and c with resistance $11/8$, hence conductance $8/11$. Combining this with the conductance of 1 on the original edge e_{ac}, we see the effective conductance between a and c is $19/11$ so the effective resistance $R_{ac} = 11/19$.

We are now prepared to prove our first theorem.

Theorem 2.1. *For $G = (V, E)$ a simple graph with $|E| = m$ and $u, v \in V$ we have*

$$C_{uv} = 2mR_{uv}.$$

Proof. For $v \in V$, recall $N(v) \subset V$ is the collection of neighbors of v and define $d(x) = |N(x)|$. Let ϕ_{uv} denote the voltage at u with respect to v in $\mathcal{N}(G)$ when $d(x)$ units of current are introduced to each vertex x and all $2m$ units of current are removed at vertex v where. Note, as G is simple every edge in $\mathcal{N}(G)$ has unit resistance. For all $u \in V \setminus \{v\}$, we observe that

$$d(u) = \sum_{w\in N(u)} i_{uw} = \sum_{w\in N(u)} (\phi_{uv} - \phi_{wv}) = d(u)\phi_{uv} - \sum_{w\in N(u)} \phi_{wv} \tag{2}$$

with the first inequality by Kirchoff's law, the second by Ohm's law with every resistance one and the third by arithmetic. Additionally, for all $u \in V \setminus \{v\}$ we see

$$H_{uv} = 1 + \sum_{w\in N(u)} \frac{H_{wv}}{d(u)} \tag{3}$$

as after one step, we have moved with probability $1/d(u)$ to a $w \in N(u)$, after which our hitting time is H_{wv}. Note that by equation (2), we have

$$d(u)\phi_{uv} = d(u) + \sum_{w \in N(u)} \phi_{wv} \Rightarrow \phi_{uv} = 1 + \sum_{w \in N(u)} \frac{\phi_{wv}}{d(u)}, \tag{4}$$

so H_{uv} and ϕ_{uv} satisfy the same system of equations and thus, by the uniqueness of such solutions, are the same.

We have now shown that H_{uv} is the voltage at u with respect to v in $N(G)$ when $d(x)$ units of current are introduced to each vertex x and all the current is removed at v. If we reverse the flow of current so that $2m$ is added at v and $d(x)$ is removed at each vertex x, we see H_{uv} is the voltage at v with respect to u.

By adding $2m$ current at u and removing $d(x)$ current at each vertex x, we get the voltage at u with respect to v to be H_{vu}. On the original network, the voltage at u with respect to v is H_{uv}. Superposing these networks, we have the voltage at u with respect to v to be the commute time C_{uv}. In doing so, the current has canceled at every vertex with the exception of $2m$ current added at u and $2m$ current removed at v. By Ohm's law, we see $C_{uv} = 2mR_{uv}$ with R_{uv} the effective resistance between u and v.

$\square$

As we saw in the proof, the commute time is essentially a measurement of voltage when the appropriate amount of current is introduced. It should also be noted that the above result can be expanded to weighted graphs quite readily. The only place we used the assumption of unit resistance on each edge was equation (2), which can be skirted by replacing $2m$ in our statement with a sum relying on the resistances of each edge.

We define the *cover time* C_x to be the expected number of steps it takes for a random walk starting at x to hit every vertex. The cover time of the graph is $C_G = \max_{x \in V} C_x$. Because the maximum commute time in some sense conveys the largest possible obstruction to covering our graph, we can use it to find bounds on commute time.

Theorem 2.2. *Let $R = \max_{u,v \in V} R_{uv}$ and $G = (V, E)$ with $|V| = n$ and $|V| = m$. Then*

$$mR \leq C_G \leq 2mR(1 + \log n).$$

Proof. Clearly $C_G \geq \max_{u,v \in V} H_{uv}$, so as $\max\{H_{uv}, H_{vu}\} \geq C_{uv}/2 = mR$ by Theorem 2.1, we have our lower bound. The upper bound follows from a result in [3] bounding C_G above by the product of the maximum hitting time H and the nth harmonic number. Since H is strictly less than the maximum commute time, we derive the upper bound by Theorem 2.1.

□

It bears mention that there is another theorem providing a second upper bound for C_G. This result requires more notation than we wish to develop at present, though the proof again reduces to using Theorem 2.1 to improve a previous result. Together, these theorems provide bounds for many classes of graphs more accurate than previously existed. In some instances, these bounds can be sharp. Additionally, the role of resistance helps clarify why situations exist where adding edges actually increases covering time.

These results merely scratch the surface of what is possible when representing random walks as electric networks. Any concept in one setting can be expressed within the language of the other. Hopefully, the reader is persuaded that this connection is fruitful.

References

[1] A. Chandra, P. Raghavan, W. Ruzzo, R. Smolensky and P. Tiwari (1996/1997), The Electrical Resistance of a Graph Captures its Commute and Cover Times, *Comput. complex.* **6**, 312–340.

[2] P. Doyle and J. L. Snell (1984) *Random Walks and Electric Networks*, The Mathematical Association of America

[3] P. Matthews (1988) Covering Problems for Brownian Motion on Spheres, *Ann. of Prob.* **16**, 189–199

[4] C. St. J. A. Nash-Williams (1959) Random walk and electric current networks *Proc. Camb. Phil. Soc.* **55** 181-194