Percolation on Oriented Square Grid with a Hint of the Fire-fighting Problem

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Abstract

First, we will briefly introduce oriented percolation with our focus on a special orientation on the square grid. Then we will discuss critical probabilities of bond or site percolation on this particular orientation. Finally, we will introduce the Fire-fighting Problem and prove a theorem regarding an upper bound for these critical probabilities in the case that fractional fire-fighters were allowed to close some open sites.

1 Preliminaries

Let Λ be a connected, infinite, and locally finite simple graph. An *orientation* on Λ is obtained by putting a direction on every bond. The resulting digraph is denoted by $\overrightarrow{\Lambda}$. Depending on whether we are considering bond or site percolation, we will assume that bonds or sites are open with probability $p \in (0,1)$. An *oriented path* is a finite or infinite path $P = x_0x_1x_2\ldots$ in $\overrightarrow{\Lambda}$ with each bond oriented from x_i to x_{i+1}. In oriented bond (respectively, site) percolation, an open path P is an oriented path with each bond (respectively, site) open. For a given site x, C_x^+ the *open out-cluster* of x is defined as the set of all sites y for which there is an open oriented path connecting x to y. Moreover, one may define $\theta_x^+(p) = \mathbb{P}_p(|C_x^+| = \infty)$ and $\chi_x^+(p) = \mathbb{E}_p(|C_x^+|)$. For clarity and simplicity, we will use "b" or "s" – depending on whether we are discussing bond or site percolation – instead of the "+" sign in the aforementioned definitions. As it is true in the non-oriented percolation, $\theta_x^c(p)$ and $\chi_x^c(p)$ – where $c \in \{b, s\}$ – are increasing functions of p and there are critical probabilities $P_{\mathbf{H}}^c(\overrightarrow{\Lambda}; x)$ and $P_{\mathbf{T}}^c(\overrightarrow{\Lambda}; x)$ such that for all $p > P_{\mathbf{H}}^c(\overrightarrow{\Lambda}; x)$ (respectively, $p > P_{\mathbf{T}}^c(\overrightarrow{\Lambda}; x)$), we have $\theta_x^c(p) > 0$ (respectively, $\chi_x^c(p) = \infty$).

2 Critical Probabilities of the Oriented Square Grid

Suppose $\overrightarrow{\mathbb{Z}^2}$ is the oriented square grid in which horizontal and vertical bonds are oriented from left to right and bottom to top, respectively. Since this digraph is site transitive, for all $x, y \in \overrightarrow{\mathbb{Z}^2}$, we have $P^c_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2}; x) = P^c_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2}; y)$ and $P^c_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2}; x) = P^c_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2}; y)$; therefore, we may only consider these critical probabilities for the origin and denote them by $P^c_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2})$ and $P^c_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2})$. Our first objective in this paper is to find a lower bound and an upper bound for these critical probabilities.

One noteworthy observation is that, since the open paths in $\overrightarrow{\mathbb{Z}^2}$ are also open paths in $\mathbb{Z}^2$, percolation in $\mathbb{Z}^2$ dominates that in $\overrightarrow{\mathbb{Z}^2}$. As a result, we have $P^c_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2}) \geq P^c_{\mathbf{H}}(\mathbb{Z}^2)$ and $P^c_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2}) \geq P^c_{\mathbf{T}}(\mathbb{Z}^2)$. Also, using the exploration method, one can show that $P^s_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2}) \geq P^b_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2})$ and $P^s_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2}) \geq P^b_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2})$. A detailed proof is given in Theorem 5 on p. 27 in [1]. Putting these observations together and assuming that $\longrightarrow$ represents the relation $\leq$, we have the following Hasse diagram:

$$\begin{tikzcd}
 & P^s_{\mathbf{H}}(\mathbb{Z}^2) \arrow[rr] & & P^s_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2}) \\
P^s_{\mathbf{T}}(\mathbb{Z}^2) \arrow[ur] \arrow[rr] & & P^s_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2}) \arrow[ur] & \\
 & P^b_{\mathbf{H}}(\mathbb{Z}^2) \arrow[uu] \arrow[rr] & & P^b_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2}) \arrow[uu] \\
P^b_{\mathbf{T}}(\mathbb{Z}^2) \arrow[uu] \arrow[ur] \arrow[rr] & & P^b_{\mathbf{T}}(\overrightarrow{\mathbb{Z}^2}) \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Based on this diagram, to find a lower bound and an upper bound for these critical probabilities, we only need to find a lower bound for $P^b_{\mathbf{T}}(\mathbb{Z}^2)$ and an upper bound for $P^s_{\mathbf{H}}(\overrightarrow{\mathbb{Z}^2})$. But we know that $P^b_{\mathbf{T}}(\mathbb{Z}^2) = 1/2$. Before proceeding with the upper bound, we will take a detour.

3 Fire-fighting on Infinite Graphs

The Fire-fighter Problem was first introduced by Hartnell [3] in 1995. For a survey of results, see [2]. Let Λ be a be a connected, infinite, and locally finite simple graph. A fire starts at a site b_0 at time $t = 0$ (time increases discretely in one unit increments). t increases by one and then on, for each time unit there are f fire-fighters (f is a fixed positive integer) on reserve to protect sites in Λ not yet affected by the fire. When protected by a firefighter, a site remains protected for rest of this process. At $t = 1$, let's assume that the sites that are protected by fire-fighters are $P'_1 = \{p_1, p_2, \ldots, p_f\}$ where p_i is distinct from b_0. After deploying these fire-fighters, the fire spreads to any unprotected neighbors of b_0 and then, t increases by one. When burned, a site remains burned for the rest of this process. We will use the pair (P_k, B_k) to represent the protected and the burned sites, respectively, by time $t = k$. Recursively, we have $P_k = P_{k-1} \cup P'_k$ and $B_k = B_{k-1} \cup (N(B_{k-1}) - P_k)$ where $B_0 = \{b_0\}$, $P_0 = \varnothing$ and P'_k is the set of previously unprotected and unburned sites that one will protect at $t = k$. In case of $\overrightarrow{\Lambda}$ an oriented graph, fire spreads only through out-bonds of a site.

In the above setting, one assumes that f is a fixed positive integer, but one can also assume that f is a fraction a/b with $0 < a < b$ and one will deploy a fire-fighters after every b time lapses.

4 Main Theorem

Theorem 1. *Let $f \in (0, 1/2)$ be a fixed fraction. Let fire starts at the origin in $\overrightarrow{\mathbb{Z}^2}$ with sites open with probability $p > f$. Consider sites that are closed, protected from the fire by default while open sites can be impacted if not protected by fire-fighters. If $p > 1 - (1/9)^{(1/2-f)^{-1}}$, the fire can not be contained.*

Before we start with the proof of the above theorem, we need to state the following theorem whose proof can be found in [4].

Theorem 2. *(Chernoff-Hoeffding) Let $X_1, X_2, \ldots, X_n$ be i.i.d with $X_i \in \{0, 1\}$ and $p = \mathbb{E}[X_i]$ for $i = 1, 2, \ldots, n$. Then for $\epsilon > 0$*

$$\mathbb{P}[\frac{1}{n}\sum_{i=1}^{n} X_i \geq p + \epsilon] \leq \big((\frac{p}{p+\epsilon})^{p+\epsilon}(\frac{1-p}{1-p-\epsilon})^{1-p-\epsilon}\big)^n \quad (\dagger)$$

and

$$\mathbb{P}[\frac{1}{n}\sum_{i=1}^{n} X_i \leq p - \epsilon] \leq \big((\frac{p}{p-\epsilon})^{p-\epsilon}(\frac{1-p}{1-p+\epsilon})^{1-p+\epsilon}\big)^n. \quad (\ddagger)$$

Note that (‡) can be derived from (†) by using $Y_i = 1 - X_i$ instead of X_i. Define D as

$$D(x, y) = x \ln(\frac{x}{y}) + (1 - x) \ln(\frac{1 - x}{1 - y})$$

which simplifies the right-hand side of (†) and (‡) to $\exp[-D(p+\epsilon, p)]^n$ and $\exp[-D(p-\epsilon, p)]^n$, respectively. Moreover, using calculus, one can show that for $x \in (0, 1/2)$ fixed and $y \in [x, 1)$, $D(x, y)$ is a strictly increasing non-negative function. Consequently, $\exp[-D(x, y)] \leq 1$.

Proof of Theorem 1: Let C_0 be the set of sites that can be reached from the origin by open paths. So if we do not deploy any firefighters, fire will reach these sites eventually, depending on their distance from the origin. Suppose that the fire is contained and as a result, C_0 is finite. So there exists an external boundary ∂C_0^{∞} blocking the fire from getting from the inside to the outside of this boundary.

Let S be any cycle of length $2l$ in $\mathbb{Z}^{2*}$ that we traverse counter-clockwise. Suppose S is a blocking cycle. There are four observations to make: 1) the bonds in S that are blocking the fire are those who are oriented upward if vertical or leftward if horizontal and this is due to the orientation on $\overrightarrow{\mathbb{Z}^2}$ and how fire spreads; 2) the number of upward or leftward bonds in S is equal to the number of downward or rightward bonds. This means that as we traverse S, we are making a total of l upward or leftward steps; 3) for S to be blocking, any upward or leftward bond in S must have a protected site to its right; 4) any site can be on the right of at most one upward and at most one leftward step. Putting these observations together, for S to be blocking, there are at least $l/2$ protected sites. Moreover, the distance from the origin to any site $x \in Int(S)$ is at most l which implies that the fire must be contained by $t = l$. Consequently, the total number of open sites that were protected by fire-fighters is at most fl. As a result, at most $l/2 - fl$ of these sites were closed and hence, protected by default. On the other hand, there are at most l sites to the right of an upward or leftward step in S. We want to show that the probability of the event A : (at most fl of these sites are open) is exponentially small. Since each site is open independently of other sites with probability p and having a Bernoulli distribution, by letting $f = p - \epsilon$ and $n = l$ in (‡), we have $\mathbb{P}[A] \leq \exp[-D(f, p)]^l$. It follows that the probability that S is a blocking cycle is bounded from above by

$$(1 - p)^{l/2 - lf} \exp[-D(f, p)]^l \leq (1 - p)^{l/2 - lf},$$

since $f \in (0, 1/2)$ is fixed and $p \in [f, 1)$.

Let L_k be the line segment joining the origin to the point $(k,0)$ and let Y_k be the number of blocking cycles around L_k. We know that the number of cycles surrounding L_k of length $2l$ is bounded from above by $4\times 3^{2l-2}$ and each is blocking with probability at most $(1-p)^{l/2-lf}$. As a result, for $p > 1-(1/9)^{(1/2-f)^{-1}}$,

$$\mathbb{E}[Y_k] \leq \sum_{l\geq k+2} l(4\times 3^{2l-2})(1-p)^{l/2-lf} \leq \sum_{l\geq k+2} \frac{4}{9} l(3(1-p)^{\frac{1}{2}(\frac{1}{2}-f)})^{2l}$$

converges and for l large enough, this expected value will be less than one. Since Y_k is a counting random variable, then $\mathbb{P}[Y_k=0]>0$. Define A_k to be the event that $Y_k=0$ and define B_k be event that $k+1$ sites in L_k are unprotected. These two events are independent, since they are disjoint. Moreover, if they both hold, there will be no blocking cycles and the fire will continue spreading indefinitely with positive probability. A_k and B_k happen with positive probability; hence, the result.□

Theorem 3. $P^s_{\boldsymbol{H}}(\overrightarrow{\mathbb{Z}^2}) \leq 80/81$.

Proof: Repeat the previous proof with $f=0$. In this case, one need not to consider the event A. Since $p>80/81$ is arbitrary, we have the result.□

References

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[2] S. Finbow, G. MacGillivray, *The Firefighter Problem: A survey of results, directions and questions*, manuscript, 2007.

[3] B. L. Hartnell, *Firefighter! An Application of Domination*, presentation at the Twentieth Conference on Numerical Mathematics and Computing, University of Manitoba, 1995.

[4] W. Hoeffding, *Probability Inequalities for Sums of Bounded Random Variables*, Journal of the American Statistical Association, Vol. 58, No. 301 (Mar., 1963), pp. 13-30.