The Complex Analysis Behind Smirnov's Theorem

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Abstract

Smirnov's celebrated proof [6] of the conformal invariance of critical site percolation on the triangular grid relies on a number of fundamental results from classical complex analysis. The purpose of this paper is to present these results in a self-contained format that any non-analysts who are interested in Smirnov's Theorem will find accessible.

1 Introduction

The usual convention for modeling two-dimensional percolation problems is to study them on a lattice embedded in $\mathbb{R}^2$. However, embedding the lattice in $\mathbb{C}$ instead offers several potential advantages, especially with regards to the question of conformal invariance. First, the requirements for a function $f : \mathbb{C} \to \mathbb{C}$ to be differentiable are stricter than the requirements for real differentiability; this restriction means that complex-differentiable (or *holomorphic*) functions have more desirable properties than is necessarily the case for real-differentiable functions. Because of this tractability, conformal maps can be characterized quite neatly in terms of holomorphic functions.

Why are we concerned with conformal maps? The "Conformal Invariance Conjecture" [5] states that if $D \subseteq \mathbb{R}^2$, and Λ is any lattice satisfying certain symmetry conditions, then if we let $P_\delta(D)$ denote the probability that, in critical percolation on Λ_δ (the lattice Λ with bond-length δ), there exists an open path traversing D, the limit

$$P(D) = \lim_{\delta \to 0} P_\delta(D)$$

exists. Furthermore, if $f : D \to D'$ is a conformal map, it is conjectured that

$$P(D) = P(D'),$$

and that these probabilities are independent of the choice of lattice Λ.

The first step towards proving the Conformal Invariance conjecture was taken by Smirnov, who shows [6] that if $D, D' \subseteq C$ are bounded and simply connected, then the limiting probabilities $P(D), P(D')$ exist for site percolation on the triangular lattice, and $P(D) = P(D')$. In this paper we present the theorems from complex analysis underlying Smirnov's Theorem. In particular, we show that analytic and holomorphic are equivalent characterizations of complex functions, and that nearly all holomorphic functions are conformal. To enable the reader to make use of this equivalence, we then present two common ways to prove that a function is holomorphic. Finally, we state but do not prove the Riemann Mapping Theorem and Carathéodory's Theorem.

2 Holomorphic and Analytic Functions

We say that a map is *differentiable* if it can be approximated locally by a linear transformation. In the case of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, f is differentiable at $\vec{v}$ if we can find a 2×2 matrix $A_{\vec{v}}$ such that

$$\lim_{||\vec{h}|| \to 0} \frac{f(\vec{v} + \vec{h}) - f(\vec{v}) - A_{\vec{v}}(\vec{h})}{||\vec{h}||} = 0. \tag{1}$$

If f is differentiable, and we write

$$f(\vec{v}) = f(v_1, v_2) = (f_1(v_1, v_2), f_2(v_1, v_2)\,),$$

then

$$A_{\vec{v}} = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} \end{bmatrix}. \tag{2}$$

If we think about f as a complex function $f : \mathbb{C} \to \mathbb{C}$, and $\vec{v} = (v_1, v_2)$ as a complex number $v = v_1 + iv_2$, then we declare f to be *holomorphic* at v if the matrix $A_{\vec{v}}$ corresponds to multiplication by a complex number. Since

$$(a + ib)(c + id) = ac - bd + i(ad + bc),$$

translating our complex numbers $c+id$ and $ac-bd+i(ad+bc)$ into vectors shows us that the matrix for multiplication by $a+ib$ is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. In other words, the requirement that $A_{\vec{v}}$ represent multiplication by a complex number forces

$$\frac{\partial f_1}{\partial v_1} = \frac{\partial f_2}{\partial v_2}; \quad \frac{\partial f_1}{\partial v_2} = -\frac{\partial f_2}{\partial v_1}, \tag{3}$$

the *Cauchy-Riemann equations.*

Not only are these equalities necessary, they are also sufficient.

Theorem 2.1 (Cauchy-Riemann)**.** *A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic iff its component functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are real-differentiable and satisfy* (3)*.*

Instead of using 1 and i as our basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$, we could use 1 and $\omega = \frac{\sqrt{-3}-1}{2}$, a cube root of unity. (This direction was pursued in detail in class on March 2.) Starting from this basis, we can develop a modified form of the Cauchy-Riemann equations that also characterizes holomorphic functions. The relationship between angles on the triangular grid Λ and the angle between the basis vectors makes it easy to show, via these modified Cauchy-Riemann equations, that the limit functions f^i measuring the crossing probabilities are holomorphic.

One can check that a function f is holomorphic by showing that it satisfies the Cauchy-Riemann equations; however, when the function's input and output are represented in the form $z = re^{it}$ rather than $z = x+iy$, understanding the component functions f_1, f_2 and their derivatives can be complicated. In these cases, to show that f is holomorphic it's often easier to use *Taylor's Theorem*, which says that all holomorphic functions are complex-analytic.

Definition A function f is *complex-analytic* on an open set $D \subseteq \mathbb{C}$ if f can be locally represented by an (infinite) power series with coefficients in $\mathbb{C}$. Formally, $\forall\ z_0 \in D,\ \exists\ r > 0$ such that if $|z_0 - z| < r$, then $\exists\ \{c_n\}_{n\in\mathbb{N}_0} \subseteq \mathbb{C}$ such that

$$f(z) = \sum_{n\geq 0} c_0(z-z_0)^n.$$

Straightforward calculations show, using the Cauchy-Riemann equations, that any complex-analytic function is holomorphic at every point $z_0 \in D$, so we see that "differentiable" and "analytic" are synonymous when applied to functions $f : \mathbb{C} \to \mathbb{C}$.

The proof of Taylor's Theorem relies on the Cauchy-Goursat Theorem, which states that if C is any simple[1] closed curve in D such that $Int(C) \subseteq D$, and f is holomorphic on D, then

$$\int_C f(z)dz = 0.$$

Morera's Theorem asserts that the converse also holds.

Theorem 2.2 (Morera). *If $\int_C f(z)dz = 0$ for every closed curve C such that C and $Int(C) \subseteq D$, then f is holomorphic on D.*

The proof of Morera's Theorem rests on constructing a holomorphic antiderivative for f; if $f = F'$ for some holomorphic F, then since F is analytic by Taylor's Theorem, so is f.

Fixing a point $z_0 \in D$, we define

$$F(z) = \int_{[z_0,z]} f(w)dw.$$

One must then check that F is well defined and holomorphic and that $F'(z) = f(z)$; for the details, see [1] or [4], for example.

3 Conformal Maps and Domains

Definition We say that a map f (on $\mathbb{C}$ or $\mathbb{R}^2$) is *conformal* if it preserves angles. More formally, f is conformal at p if for any two curves $\alpha, \beta : (-\epsilon, \epsilon) \to \mathbb{C}$ such that

$$\alpha(0) = \beta(0) = p, \quad \alpha'(0) \neq 0,\ \beta'(0) \neq 0, \tag{4}$$

we have

$$\frac{(f \circ \alpha)'(0)}{(f \circ \beta)'(0)} = \frac{\alpha'(0)}{\beta'(0)}. \tag{5}$$

[1]non-self-intersecting

To see the equivalence of these two definitions, write

$$\frac{(f\circ\alpha)'(0)}{(f\circ\beta)'(0)} = \frac{r_1 e^{it_1}}{r_2 e^{it_2}} = \frac{r_1}{r_2} e^{i(t_1-t_2)}.$$

This format of writing a complex number – in terms of its radial distance r from the origin and the angle t its vector makes with the positive real axis – shows us that the *argument*, the variable t, of $\frac{(f\circ\alpha)'(0)}{(f\circ\beta)'(0)}$ measures the angle between the two vectors (or complex numbers) $(f\circ\alpha)'(0)$ and $(f\circ\beta)'(0)$. Thus, the condition (5) means that f preserves angles and ratios of moduli.

Of course, to satisfy (5), the function f must be holomorphic at p. In fact, this necessary condition is very nearly sufficient.

Proposition 3.1. *If f is holomorphic at p, and $f'(p) \neq 0$, then f is conformal at p.*

Proof: By the chain rule, if $\alpha : (-\epsilon, \epsilon) \to \mathbb{C}$, then $(f\circ\alpha)'(0) = f'(\alpha(0))\alpha'(0)$. Thus, if $f'(p) \neq 0$, and α, β satisfy (4), then (5) follows immediately. □

Smirnov's proof that the limiting crossing probabilities are invariant under conformal maps begins by defining certain functions f^i_δ on the vertices of the triangular grid Λ_δ underlying the domain D, and showing that they measure the probability of an open path crossing D in the i direction. He proceeds to show that for each i, the sequence $\{f^i_\delta\}_\delta$ converges, as $\delta \to 0$, to a holomorphic (and hence conformal) map f^i. Bollobás and Riordan [2], following Smirnov's original approach, prove this using Morera's Theorem, rather than the modified Cauchy-Riemann equations we discussed in class. Finally, a uniqueness condition guarantees that if $\phi : D \to D'$ is a conformal map, then $f^i \circ \phi$ measures the limiting crossing probability on D'. In other words, once we know the crossing probabilities on D, we also know them for any domain D' which is holomorphic to D. (See [2] for the details of the proof.)

Now that we have characterized conformal maps in terms of holomorphic maps, a natural next question is to study which subsets of $\mathbb{C}$ have conformal maps between them. Because the proofs of the following results have simple, elegant statements and long, convoluted proofs, we present merely the statements here, referring the reader to [1] for the details.

Theorem 3.2 (Riemann Mapping Theorem)**.** *If D is any simply connected proper open subset of $\mathbb{C}$, then there exists a conformal bijection ϕ between D and the open unit disc $B_1(0)$.*

In other words, we have a conformal map between any two simply connected, proper, open subsets of $\mathbb{C}$.

Theorem 3.3 (Carathéodory)**.** *If $\phi : D \to B_1(0)$ is a conformal map, and ∂D is a simple closed curve, then ϕ extends to a homeomorphism $\hat{\phi} : \overline{D} \to \overline{B_1(0)}$.*

These theorems are really what enable us to reap the benefits of conformal invariance; since all reasonable domains are conformally equivalent, once we know the crossing probabilities for one such domain, Smirnov's Theorem combines with the Riemann Mapping Theorem to tell us the crossing probabilities for all of them. Moreover, years before Smirnov's proof, Cardy [3] and Carleson calculated what these probabilities should be in the case of an equilateral triangle with vertices at 1, ω, and ω^2, and the proof of Smirnov's Theorem confirms their formula. Thus, we can calculate explicitly the crossing probabilities for any domain $D \subseteq \mathbb{C}$ whose boundary is a simple closed curve.

References

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