

Lilypad Percolation

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Abstract

Let r be a nonnegative real number. Attach a disc of radius r to infinitely many random points (including the origin). Lilypad percolation asks whether we can reach infinity from the origin by walking through ‘lilypads’, that is, moving from one disc to another only if the discs overlap. In this paper, we explain what we mean by infinitely many random points, giving a definition of a Poisson Random Process. We also prove a theorem showing some lower bounds for r where we percolate and some upper bounds for r where we are stuck in finite land. We end with some remarks on other developments and numerical results, connecting lattice percolation models with continuum percolation models.

1 Introduction

Given a real positive number λ , we pick infinitely many points (including the origin) randomly with density λ , that is, in a figure U of area $|U|$ the expected number of points is $\lambda \cdot |U|$ (λ points per unit area). The way we achieve this is by doing a Poisson process, which I shall describe in Section 2. What a Poisson process does is give us a way to pick infinitely many points randomly in a way where we get nice independence conditions.

Now that we have an infinite set of random points, given a real positive r , we attach a disc of radius r to each point. Now, we’d like to know if with positive probability we can reach infinity from the origin by walking through the discs (lilypads). In other words, the question is whether we can percolate. In contrast to the case with percolation on the grid, where there is a critical probability, in this problem we have a critical area, since making it to infinity depends on r and λ , not on some probability. Let $D_{r,\lambda}$ be the graph where the vertices are the random points we get through a random process of density λ , and where two points have an edge iff their discs overlap. This would be the lilypad graph.

Consider the graph $G_{r,\lambda}$ defined by picking infinite random points with density λ to be the vertices and drawing an edge if the distance between two points is $\leq r$. The question of whether $G_{r,\lambda}$ percolates is very similar to the lilypad model. In fact $G_{2r,\lambda}$ percolates iff $D_{r,\lambda}$ percolates. The graphs are not isomorphic because of possible overlaps, but the question of percolation is identical, hence studying $G_{r,\lambda}$ is the way to go.

Note that the degree of a vertex in $G_{r,\lambda}$ is the number of points inside the disc of radius r , which has area $\pi r^2 \lambda$, therefore the expected degree of a vertex is $\pi r^2 \lambda$. The structure of $G_{r,\lambda}$ depends only on the degree, so we can think of $G(a)$ for any graph $G_{r,\lambda}$ where $a = \pi r^2 \lambda$ is constant.

The critical degree a_c is the number such that for all $a < a_c$ $G(a)$ does not percolate and for $a > a_c$, $G(a)$ percolates with positive probability. In section 3, I will prove the following theorem:

Theorem 1.1 (Hall).

$$\frac{2\pi \log 2}{3\sqrt{3}} \leq a_c \leq \frac{26\pi \log 2}{3\sqrt{3}}.$$

In Section 4, we'll do a slight improvement of the upper bound due to Hall [5]. In section 5 we'll use a different technique to get a better lowerbound due to Gilbert (in fact, lilypad percolation is also known as Gilbert percolation due to his 1961 seminal paper [2]) . This lower bound uses very different techniques. In Section 6 we'll mention several other results without proof.

2 Poisson Process

Let λ be a positive real number, and let $P_\lambda \in \mathbb{R}^2$ be a random countably infinite set of points in the plane. Let $\mu(U)$ be the the number of points P_λ in a bounded Borel set ¹. P_λ is a homogeneous Poisson process of density λ if for n pairwise disjoint Borel sets U_i , $\mu_\lambda(U_i)$ are independent and for every bounded Borel set U , $\mu_\lambda(U)$ is a Poisson random variable with mean $\lambda|U|$ where $|U|$ is the standard measure of U .

¹In case, you're wondering, a Borel set is the union or intersection of a bunch of open,or closed sets. For generality, we want to use Borel sets, but in Gilbert percolation we're mostly interested in nice sets like figures such as hexagons

One possible construction (which can be shown turns out to be the only such random process) is the following:

For $\lambda > 0$, let $\{X_{i,j} : (i,j) \in \mathbb{Z}^2\}$ be independent Poisson random variables, each with mean λ . Thus,

$$\mathbb{P}(X_{i,j} = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

for $k = 0, 1, 2, \dots$. Let $Q_{i,j}$ be the unit square with bottom left vertex $(i,j) \in \mathbb{Z}^2$, select $X_{i,j}$ points independently and uniformly from $Q_{i,j}$, then the union of all these sets satisfies the properties needed to be a Poisson process, so we'll use this as our definition of P_λ .

3 Critical Area: The Easy Bounds

Proof of Main Theorem. Consider a hexagon lattice with side s . Consider a face to be open if it contains a point from P_λ and closed otherwise. A face is closed with probability $\mathbb{P}(X = 0) = e^{-\lambda A}$ where A is the area of the hexagonal face, that is $A = \frac{3\sqrt{3}}{2}s^2$. Therefore, a face is open with probability $1 - e^{-\lambda A}$. Now, face percolation on the hexagon lattice is equivalent to site percolation on the triangular lattice which has critical probability $p = \frac{1}{2}$. Since $p = 1 - e^{-\lambda A}$, $e^{-\lambda A} = \frac{1}{2}$, therefore $\lambda A = \log 2$. Letting $\lambda = 1$ we have $A = \log 2$, hence $s^2 = \frac{2\log 2}{3\sqrt{3}}$. Giving us the critical threshold with respect to s for face percolation on the hexagonal lattice.

Now, we have to relate this back to the disc percolation. When trying to find a lowerbound, we want to fail at percolation, hence we want $s^2 < \frac{2\log 2}{3\sqrt{3}}$. The hexagonal lattice failing is not enough to make the disc fail, because we could have two points in the same disc while being in non-adjacent hexagonal faces. To force the failure of hexagonal percolation to imply lilypad failure, we need to have $r < s$ since s is the closest distance between two non-adjacent hexagonal faces. Then, we have $a = \pi r^2 < \pi s^2 < \frac{2\pi \log 2}{3\sqrt{3}}$. Since $\lambda = 1$, we have that for any $a < \frac{2\pi \log 2}{3\sqrt{3}}$ percolation fails, giving us $a_c \geq \frac{2\pi \log 2}{3\sqrt{3}}$.

Let's do the upper bound. In this case we want to percolate, so we want $s^2 > \frac{2\log 2}{3\sqrt{3}}$. The hexagonal lattice percolating is not enough to guarantee that the disc percolates; we need to guarantee that any two points in adjacent hexagonal faces are contained in the same disc. Using law of cosines and the Pythagorean theorem we can show that the distance between the two farthest apart points between two adjacent hexagonal faces is $\sqrt{13}s$. Therefore we want

$r > \sqrt{13}s$, giving us that $a = \pi r^2 > 13\pi s^2 > \frac{26\pi \log 2}{3\sqrt{3}}$, showing that $a_c \leq \frac{26\pi \log 2}{3\sqrt{3}}$. \square

The theorem shows numerically that $0.838153\dots \leq a_c \leq 10.896\dots$

I would like to point out that since for lilypad percolation we care about r in $G_{2r,\lambda}$, we could show that the critical radius is between $0.25826\dots$ and $0.931169\dots$. This shows that if $r = 1$ then the radius is big enough for a path to infinity to happen among the lilypads.

4 Critical Area: Improving the upper bound

Theorem 4.1 (Hall).

$$a_c \leq \frac{12\pi \log 2}{3\pi - 18 \arcsin\left(\frac{1}{4}\right) - \frac{9\sqrt{3}(\sqrt{5}-1)}{8}} = 10.588\dots$$

Proof. The idea of the proof is to keep using the hexagonal lattice, but now instead of making a face open if there is a point from P_λ in the hexagon, we will make it open if there is one in the rounded hexagon. The rounded hexagon is built as follows: take an hexagon of side s , now from every midpoint of a segment draw a circle of radius $\sqrt{3}s$ which is the distance to the opposite midpoint (by Pythagoras). The rounded hexagon consists of the figure that remains in the middle, it has six vertices but the edges are rounded instead of straight.

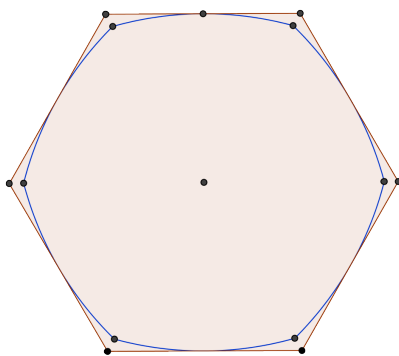


Figure 1: A Rounded Hexagon.

Since the area of the rounded hexagon is smaller, it seems that we give away our chances to percolate, however now the distance between two points from two adjacent hexagons is

²You may notice that the proof depends on percolation on the hexagonal lattice which had not been proved in Gilbert's time (1961).

bounded by $\sqrt{12}s$ since the largest distance between any two points inside a rounded hexagon is $\sqrt{3}s$ (by our construction), so for adjacent hexagons we get $2\sqrt{3}s = \sqrt{12}s$. Hence to percolate we want $r > \sqrt{12}s$ and $A > \log 2$ where A is the area of the rounded hexagon.

Let's find the area of the rounded hexagon. Following Figure 2, let \mathbf{O} be the center of the hexagon, let \mathbf{P} be a midpoint of a segment of the hexagon and \mathbf{Q} the opposite midpoint. Let \mathbf{S} be a vertex of the hexagon as shown in Figure 2. Let \mathbf{T} be the intersection of the circle with center in \mathbf{P} and radius $\sqrt{3}s$ with $\overline{\mathbf{OS}}$, that is a vertex in the rounded hexagon. Notice that the area of the rounded hexagon is twelve times the area of (\mathbf{OQT}) . To calculate (\mathbf{OQT}) we will calculate (\mathbf{PQT}) and subtract $(\triangle \mathbf{POT})$.³

The area of (\mathbf{PQT}) is easy, as it is a portion of the circle of radius $\sqrt{3}s$. If we let α be the angle $\angle \mathbf{TPQ}$ and $\beta = \angle \mathbf{PTO}$ then $(\mathbf{PQT}) = \frac{1}{2}\alpha(\sqrt{3}s)^2 = \frac{3}{2}\alpha s^2$.

To calculate the area $\triangle \mathbf{POT}$ we will need to play around with angles and lengths. $\overline{\mathbf{PO}} = \frac{\sqrt{3}}{2}s$ because it is half the radius and $\overline{\mathbf{PT}} = \sqrt{3}s$ because it is the radius. $\angle \mathbf{POT} = 150$ degrees. Using that $\sin(150) = \frac{1}{2}$ and law of sines on $\triangle \mathbf{POT}$ we find that $\sin \beta = \frac{1}{4}$ and $\cos \beta = \frac{\sqrt{15}}{4}$. Now $\sin \alpha = \sin(30 - \beta) = \sin 30 \cos \beta - \sin \beta \cos 30 = \frac{\sqrt{15}}{8} - \frac{\sqrt{3}}{8} = \frac{\sqrt{3}}{8}(\sqrt{5} - 1)$. Therefore $(\triangle \mathbf{POT}) = \frac{\overline{\mathbf{PT}} \cdot \overline{\mathbf{PO}} \cdot \sin \alpha}{2} = \frac{3\sqrt{3}(\sqrt{5}-1)}{32}s^2$.

Using that $\alpha = \frac{\pi}{6} - \beta$ we see that $\alpha = \frac{\pi}{6} - \arcsin(\frac{1}{4})$. Using this in (\mathbf{PQT}) and plugging in

³I write parenthesis to mean area. If a triangle is not displayed to the left of the letters the figure has at least one segment that is not a straight line.

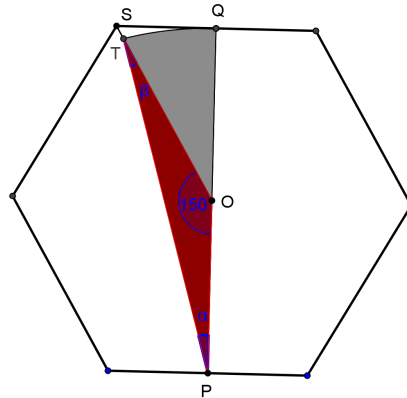


Figure 2: The red area is $(\triangle \mathbf{POT})$ and the gray area is (\mathbf{OQT})

(**Δ POT**) and multiplying by twelve we get that the area of the rounded hexagon is

$$A = \left(3\pi - 18 \arcsin\left(\frac{1}{4}\right) - \frac{9\sqrt{3}(\sqrt{5}-1)}{8} \right) s^2.$$

Combining that $A > \log 2$ and that $r > \sqrt{12}s$, we find the upper bound we were looking for. \square

5 Critical Area: Improving the lower bound

Theorem 5.1 (Gilbert).

$$\frac{6\pi}{2\pi + 3\sqrt{3}} = 1.642\dots \leq a_c.^4$$

Proof. Let C_0 be the vertex set of the component of the origin in $G_r = G_{r,1}$. We shall use an algorithm to find the points of C_0 one by one. We will want a sequence of disjoint sets of points (D_i, L_i) . To start the sequence $D_0 = \emptyset$ and $L_0 = \{X_0\} = 0$. Next, let N_0 be the set of neighbors of X_0 , and set $D_1 = \{X_0\}$ and $L_1 = N_0$. If $L_1 = \emptyset$ then we're done and $C_0 = D_1 = \{X_0\}$ finite. If $L_1 \neq \emptyset$ then pick $X_1 \in L_1$ and define $D_2 = D_1 \cup \{X_1\} = \{X_0, X_1\}$. Now let N_1 be the set of neighbors of X_1 that are not neighbors of any of the points in D_1 (that is, not neighbors of X_0). $L_2 = N_1 \cup L_1 - \{X_1\}$. We keep on doing this process, to be more specific, if we have up to (D_t, L_t) , then if $L_t = \emptyset$ then we are done, otherwise, pick $X_t \in L_t$ and define $D_{t+1} = D_t \cup \{X_t\} = \{X_0, X_1, \dots, X_t\}$. Let N_t be the set of neighbors of X_t that are not neighbors of any of the points D_t . Now $L_{t+1} = N_t \cup L_t - \{X_t\}$.

By the way we're building these sets, we can see that D_t and L_t are disjoint and $D_t \cup L_t \subset C_0$. Also, since any disc of radius r contains finitely many points, D_t and L_t are finite. Furthermore, if $L_t = \emptyset$, then $D_t = C_0$, which would imply the origin does not percolate.

$$D_t - \{X_0\} = \{X_1, X_2, \dots, X_{t-1}\} \subset \bigcup_{i=0}^{t-2} N_i, \text{ implying that } |D_t| - 1 = t - 1 \leq \sum_{i=0}^{t-2} |N_i|.$$

Let V_t be the disc of radius r with centre X_t , and set $U_t = \bigcup_{s=0}^t V_s$ (the area of all the discs so far). Conditioning on the points X_0, X_1, \dots, X_t , $|N_t|$ is a Poisson random variable with mean $|V_t - U_{t-1}|$. What is going on, is that the set of neighbors that aren't neighbors of things in D_t has area $V_t - U_{t-1}$ and since it is a Poisson process we have that $|N_t|$ is a random variable.

⁴Gilbert in fact showed a better lower bound, $1.75\dots$ in his 1961 paper [2]. It is worthwhile noting that the proof does not depend on percolation on the hexagonal lattice like the previous proofs.

As you can picture if you do the process by hand, the area the neighbors can occupy is not big, in fact we can put a trivial upper bound, to calculate it we can picture two points r units apart and consider the area of the circle of radius r minus the area of the overlap.

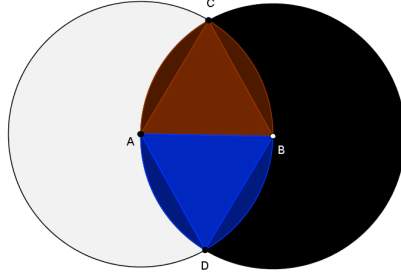


Figure 3: The area in black, which we'll call b , is the biggest $|N_t|$ could be.

Let's calculate this area. Name the points **A** and **B** as in Figure 3. Now, the two circles intersect in points **C** and **D**. Let's consider the upper half of the overlap (the red area). We can calculate it by noting that it is two times a sixth of the circle minus the equilateral triangle⁵, therefore the area of the overlap is $\left(\frac{2}{3}\pi - \frac{\sqrt{3}}{2}\right)r^2$. Since the area of the circle is πr^2 , we have that the area we're looking for is $\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)r^2 = b$.

To bound $|C_0|$, let Z_0, Z_1, \dots be independent Poisson random variables with $\mathbf{E}(Z_0) = \pi r^2$ and $\mathbf{E}(Z_i) = b$ for $i \geq 1$. Then,

$$\mathbb{P}(|C_0| \geq k) = \mathbb{P}(|D_k| = k) \leq \mathbb{P}\left(\sum_{i=0}^{k-2} |N_i| \geq k-1\right) \leq \mathbb{P}\left(\sum_{i=0}^{k-2} Z_i \geq k-1\right).$$

If $b < 1$, then $\mathbb{P}(|C_0| \geq k) \rightarrow 0$ as $k \rightarrow \infty$, hence it doesn't percolate.

$$b < 1 \text{ implies } r^2 < \frac{1}{\frac{\pi}{3} + \frac{\sqrt{3}}{2}} = \frac{6}{2\pi + 3\sqrt{3}} \text{ hence } a = \pi r^2 < \frac{6\pi}{2\pi + 3\sqrt{3}},$$

giving us the desired lower bound for the critical area.

□

Using a multi-branching process, Hall [5] was able to get a better lower bound, namely 2.184.

⁵If you pick the slice of the left circle cut by **AC** and **AB** and you sum the slice of the right circle cut by **BC** and **BA** you'll get the overlap plus the equilateral triangle, which is why you need to subtract it afterwards. A slice of the circle is a sixth of the area of the circle, hence $\frac{\pi r^2}{6}$. The equilateral triangle has area $\frac{\sqrt{3}}{4}r^2$.

6 Further Work

Many people have worked on simulations to get numerical estimates for the critical area. Quintanilla, Torquato and Ziff [3] give a lower bound of 4.51218 and an upper bound of 4.51228, putting the radius of the lilypad between .599223 and .599229, so about .6.

Using Monte Carlo methods, Balister, Bollobás and Walters [4] show that, with confidence 99.99%, the critical degree is between 4.508 and 4.515.

Several results for percolation on lattices have analogues for continuum percolation (Gilbert percolation, lilypad). I will end the paper with one result of this kind due to Roy [6] giving us an analogue of $p_H = p_T$ for lilypad percolation. The theorem is the following:

Theorem 6.1. *Let $\pi r^2 \lambda = a < a_c$, let $|C_0(G_{r,\lambda})|$ denote the number of points in the component of the origin in $G_{r,\lambda}$. Then*

$$\mathbb{P}(|C_0(G_{r,\lambda})| \geq n) \leq \exp(-c_a n),$$

where $c_a > 0$ does not depend on n . In particular, $a_T = a_c$, where

$$a_T = \inf\{\pi r^2 \lambda : \mathbf{E}(|C_0(G_{r,\lambda})|) = \infty\}.$$

References

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