The Aizenman-Kesten-Newman Theorem

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Abstract

The uniqueness of the infinite open cluster in the setting of bond percolation on the square grid was proven by Harris in 1960 [6]. As shown by Fisher in 1961 [4], Harris' proof can be extended to include site percolation on the square grid. Aizenman, Kesten, and Newman [1] show that this fact is true in a much more general setting, as well.

Let Λ be a connected, infinite, locally-finite, vertex-transitive graph, and take $\Omega = \{0,1\}^{V(\Lambda)}$ to be the probability space. The Aizenman-Kesten-Newman theorem states that under these conditions, there can be at most one infinite open cluster. We will discuss the proof of this result discovered by Burton and Kean [2].

1 Introduction

The Aizenman-Kesten-Newman theorem, particularly when combined with Menshikov's theorem, is an exceptional tool for simplifying a great variety of proofs. For instance, it allows for a simpler proof of the Harris-Kesten theorem, i.e. $p_H = p_T = 1/2$. [3]. It can also be used to prove that for any planar lattice Λ (satisfying some symmetry conditions), $p_c^b(\Lambda) + p_c^b(\Lambda^*) = 1$, where Λ^* is Λ's dual lattice. There are many more such examples (see chapter 5 of Bollobás and Riordan [3]), but to see one example of how this theorem may be applied, let us consider the proof of the fact that for the triangular lattice T, $p_c^s(T) = 1/2$.

Suppose for contradiction that $p_c^s(T) < 1/2$ and consider a lattice with percolation probability $p = 1/2$. Let H_n be the origin-centered hexagon with n sites on each of the six sides, and number the sides of H_n cyclically. Define the event L_i to be "an infinite open path leaves from side i." Similarly, let $L_i^* =$ "an infinite closed path leaves from side i," and let $E = L_1 \cap L_2^* \cap L_4 \cap L_5^*$. By Harris' lemma [6], $\mathbb{P}(E) > 0$, and since E is independent of the

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event H_{n-1} (since the infinite paths only meet H_n at its boundary), we have that $\mathbb{P}(E$ occurs and H_{n-1} has all closed sites$) > 0$. Let P_1, P_2^*, P_4, and P_5^* be the corresponding paths, as in Figure 1.

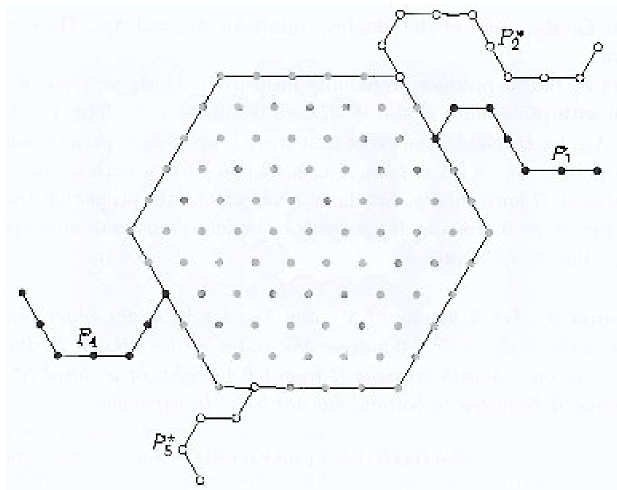

Figure 1: H_n with the paths P_1, P_4, P_2^*, and P_5^*. [3]

Then P_1 and P_4 are disconnected infinite paths, contradicting the Aizenman-Kesten-Newman theorem. Thus $p_c^s(T) = p_H^s(T) = p_T^s(T) = 1/2$.

Define I_k to be the event that there are exactly k infinite open clusters (note that, a priori, k may be infinite). To prove the uniqueness of the infinite cluster, we will need to use the result that for all k, $\mathbb{P}(I_k) \in \{0, 1\}$. Then we shall go through the proof that $\mathbb{P}(I_k) = 0$ for all $k \geq 2$, giving us our desired result.

2 Preliminary results

As mentioned, we need to use the result that $\mathbb{P}(I_k) \in \{0, 1\}$ for all k. This is true for all *automorphism invariant* events – that is, all events that are mapped into themselves by any automorphism induced by an automorphism of Λ [3]. To see that I_k is automorphism invariant, note that for any automorphism $\phi : \Lambda \to \Lambda$, two vertices x and y in Λ are adjacent if and only if $\phi(x)$ and $\phi(y)$ are adjacent in $\phi(\Lambda)$. This will give us the following result of Newman and Schulman [7].

Lemma 2.1. *For all $k \in [2, \infty)$, $\mathbb{P}(I_k) = 0$.*

Proof. Fix some vertex $x_0 \in V(\Lambda)$ and let $k \in \mathbb{N}$. Suppose that $\mathbb{P}(I_k) > 0$ (and therefore that I_k holds). Define $T_{n,k}$ to be the event that I_k holds and each infinite cluster contains a site in $B_n(x_0)$, the ball of radius n around x_0. Since these balls cover Λ, we have that for large enough n, changing each closed site in $B_n(x_0)$ to open will connect the k infinite clusters that meet $B_n(x_0)$. Therefore, $\mathbb{P}(I_1) > 0$. But then we have that $\mathbb{P}(I_k) = \mathbb{P}(I_1) = 1$, which tells us that $k = 1$. $\square$

Before we continue to Burton and Keane's theorem, we will need the following technical lemma about graphs [3].

Lemma 2.2. *Let G be a connected finite graph. Let $L = \{l_1, \dots, l_t\}$, $C = \{c_1, \dots, c_s\} \subseteq V(G)$ be disjoint, and suppose that deleting c_i disconnects G into components, at least 3 of which contain vertices of L. Then $t \geq 2 + s$.*

Proof. Note that the "worst case" is when G has the minimal amount of edges necessary to make it connected and to have $L, C \subseteq V(G)$. Hence, the case to consider is when G is a tree whose leaves are contained in L. Then for all i, $d(c_i) \geq 3$. But we know that a tree has $2 + \sum_{v \in I(V)} (d(v) - 2)$ leaves, where $I(V)$ is the set of internal vertices, so

$$|L| \geq (\text{number of leaves}) \geq 2 + \textstyle\sum_{i=1}^{s} (3 - 2) = 2 + s. \qquad \square$$

Note also that this result can be extended to disconnected graphs by considering one component at a time.

3 The uniqueness of the infinite open cluster

Define a graph Λ to be *amenable* if, for all $x \in V(\Lambda)$,

$$\lim_{n \to \infty} \frac{|S_n(x)|}{|B_n(x)|} = 0.$$

Now we are ready to tackle Burton and Keane's proof of the fact that there can be at most one infinite cluster.

Theorem 3.1. *For any infinite, connected, locally finite, amenable, vertex-transitive graph Λ, $\mathbb{P}(I_k) = 0$ for all $k \geq 3$.*

Proof. Suppose, for contradiction, that $\mathbb{P}(I_k) > 0$ for some $k \geq 3$. As before, fix a vertex $x_0 \in V(\Lambda)$. Fix r such that $B_r(x_0)$ has a positive probability of containing sites from at least 3 infinite clusters. For $x \in V(\Lambda)$, let $T_r(x) =$ "all sites in $B_r(x)$ are open and there exists an infinite cluster C such that when all sites in $B_r(x)$ are changed to closed, C is disconnected into at least 3 infinite clusters." Note that, by transitivity, for all $x \in V(\Lambda)$, $\mathbb{P}(T_r(x)) = a > 0$.

Let $W \subseteq B_{n-r}(x_0)$ be maximal subject to the constraint that the balls $B_{2r}(w)$ are disjoint for all $w \in W$. By the maximality of W, for any $z \in B_{n-r}(x_0)\backslash W$, $d(z, w) \leq 4r$ for some $w \in W$. Note that the balls $\{B_{4r}(w) \mid w \in W\}$ cover $B_{n-r}(x_0)$. Therefore

$$|W| \geq \frac{B_{n-r}(x_0)}{B_{4r}(x_0)}.$$

Using the fact that we can find a positive constant c such that $|W| \geq c|B_{n+1}(x_0)|$ for all $n \geq r$, and that Λ is amenable, we have $|W| \geq \frac{1}{a}|S_{n+1}(x_0)|$ for sufficiently large n. Let us fix such an n.

Define $B_r(w)$ to be a *cut-ball* if $w \in W \subseteq B_{n-r}(x_0)$ and $T_r(w)$ holds, i.e. $B_r(w) \subseteq B_n(x_0)$ and every site in $B_r(w)$ is open. Let s be the number of cut-balls. Then

$$\mathbb{E}(s) = \sum_{w \in W} \mathbb{P}(T_r(w)) = a|W| \geq |S_{n+1}(x_0)|.$$

Therefore, $\mathbb{P}(s \geq |S_{n+1}(x_0)|) > 0$. For the remainder of this proof, consider a configuration ω under which $s \geq |S_{n+1}(x_0)|$.

Let K be the union of all infinite clusters that meet $B_n(x_0)$. Change all of the sites in cut-balls to closed. Then K is disconnected into infinite clusters $L_1, \ldots, L_t$ and finite clusters $F_1, \ldots, F_u$. Note $t \leq |S_{n+1}(x_0)|$ since each infinite cluster contains a site in the sphere $S_{n+1}(x_0)$. Let $C_1, \ldots, C_s$ be the cut-balls.

Now recall Lemma 2.2, where the graph G is defined by contracting each C_i, F_i, and L_i to a single vertex c_i, f_i, and l_i, respectively (as in Figure 2). The infinite components of K correspond to components of G containing at least one vertex in $L = \{l_1, \ldots, l_t\}$. Note that since C_i is a cut-ball, deleting c_i from G disconnects a component into at least 3 components containing vertices of L. Applying Lemma 2.2 then says that

$$|S_{n+1}(x_0)| \geq t \geq s + 2 \geq |S_{n+1}(x_0)| + 2,$$

which is a contradiction. $\square$

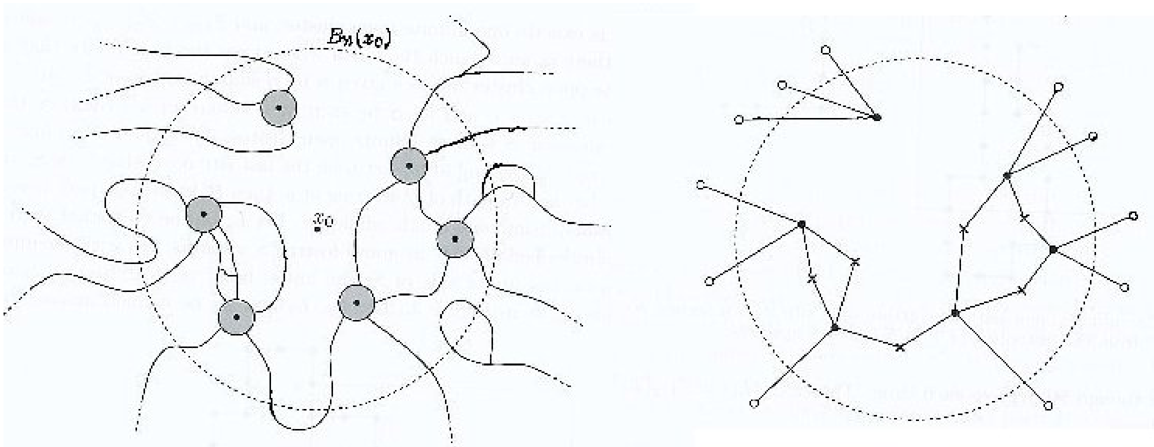

Figure 2: Constructing the graph G from K. [3]

4 A possible direction for future work

One easy verification of the Aizenman-Kesten-Newman theorem for the special case of bond percolation in $\mathbb{Z}^2$ can be constructed as follows.

Suppose, for contradiction, that there are at least two infinite open clusters, C_1 and C_2. Choose $r \in \mathbb{N}$ such that $B_r(0)$, the square of radius r centered at $(0,0)$, is the smallest (origin-centered) square that meets both C_1 and C_2. Now consider an annulus surrounding $B_r(0)$. By Harris' lemma [6], each of the four rectangles comprising the annulus is crossed length-wise by an open path with some probability $\varepsilon > 0$ as in Figure 3.

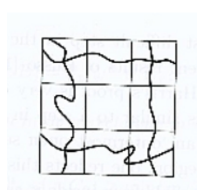

Figure 3: An annulus with open path P. [3]

If such a path P exists in each of the four rectangles, then C_1 and C_2 must be connected by an open path. Iteratively taking (proportional) annuli around the resulting square, we have (with probability 1) some annulus with a path connecting the two clusters.

Clearly this construction does not immediately generalize. However, perhaps one can find an extension of this proof to (for example) a three-dimensional hypercubic lattice, where the clusters would become surfaces.

References

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